Mechanically driven pendula for instructional laboratories

R. M. Pacheco

Follow this and additional works at: https://commons.emich.edu/theses

Part of the Physics Commons

Recommended Citation

https://commons.emich.edu/theses/931
Mechanically Driven Pendula for Instructional Laboratories

by

R.M. Pacheco

Thesis

Submitted to the school of Physics and Astronomy
Eastern Michigan University,
in partial fulfillment of the requirements

for the degree of

MASTER OF SCIENCE

in

Physics

Thesis committee:
Ernest Behringer, Ph.D.
Marshall Thomsen, Ph.D.
Dave Pawlowski, Ph.D.

November 10, 2018
Ypsilanti, Michigan
Acknowledgments

Thank you, Dr. Behringer. You truly showed me the light at the end of the proverbial tunnel. Thank you for your patience and hard work over the past six months and for the excellent thesis idea. Thank you, Dr. Thomsen. Your keen insights and helpful nature have been one of the best parts of my Eastern experience. Thank you, Dr. Pawlowski and Dr. Thomsen for joining my committee. Thank you, Hans Harff, for helping me derive damping constants, there is never a damped moment around you. Thank you Jesse Mason, for forever editing my written word. Thank you, wife. Without Molly, I am lost.
Abstract

I present a new, low-cost approach to observing driven resonance with simple and physical pendula. I mount a pendulum on a dynamics cart that is made to oscillate along a horizontal line by a stepper motor and micro-controller. The pendulum pivot therefore has a position that varies sinusoidally with time with a constant, adjustable frequency. I designed and constructed the experiment to be easily implemented into any physics lab. I tested the apparatus and observed driven resonance for both types of pendula. All of the measured resonant frequencies I determined using the apparatus had percent uncertainties under 4% and all of the predicted resonant frequencies of the pendula fell within the experimental frequency uncertainty ranges. None of the leading vendors of apparatus for instructional physics labs have pendulum attachments for dynamics carts, making this a new experimental approach for undergraduate or graduate physics students to observe a driven resonator. The cost of all the equipment, excluding dynamics carts and tracks, under $200.
Table of Contents

Acknowledgments ............................................................... ii

Abstract ........................................................................ iii

List of Tables .................................................................. vii

List of Figures .................................................................. viii

1 Introduction .....................................................................

2 Theory ........................................................................... 3
  2.1 Simple Pendulum ....................................................... 3
  2.2 Physical Pendulum ..................................................... 5

3 Design and Construction .................................................. 8
  3.1 Electronic Components ............................................... 8
  3.2 Printed Components .................................................. 10
    3.2.1 Pendulum Arm ................................................... 10
    3.2.2 Square Bracket .................................................. 11
  3.3 Pendulum Design and Hardware ................................. 13
    3.3.1 Simple Pendulum Design ................................. 13
    3.3.2 Physical Pendulum Design .............................. 14
    3.3.3 Connecting the Cart and Motor ..................... 16
  3.4 Measuring Angular Displacement Amplitude ................. 18
    3.4.1 Visual Measurements .................................... 18
    3.4.2 Camera Phones .............................................. 19
    3.4.3 UV LED ...................................................... 19

4 Experiment ....................................................................... 20
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Procedure</td>
<td>21</td>
</tr>
<tr>
<td>4.2 Data and Results</td>
<td>22</td>
</tr>
<tr>
<td>4.2.1 Simple Pendula Data and Results</td>
<td>23</td>
</tr>
<tr>
<td>4.2.2 Physical Pendula Data and Results</td>
<td>31</td>
</tr>
<tr>
<td>4.3 Calculations</td>
<td>36</td>
</tr>
<tr>
<td>5 Conclusions</td>
<td>38</td>
</tr>
<tr>
<td>References</td>
<td>44</td>
</tr>
<tr>
<td>Appendix A</td>
<td>46</td>
</tr>
<tr>
<td>Appendix B</td>
<td>48</td>
</tr>
<tr>
<td>B.1</td>
<td>48</td>
</tr>
<tr>
<td>B.2</td>
<td>49</td>
</tr>
<tr>
<td>Appendix C</td>
<td>51</td>
</tr>
<tr>
<td>Appendix D</td>
<td>53</td>
</tr>
<tr>
<td>D.1</td>
<td>53</td>
</tr>
<tr>
<td>D.2</td>
<td>55</td>
</tr>
<tr>
<td>Appendix E</td>
<td>57</td>
</tr>
<tr>
<td>Appendix F</td>
<td>62</td>
</tr>
<tr>
<td>Appendix G</td>
<td>64</td>
</tr>
<tr>
<td>G.1</td>
<td>64</td>
</tr>
<tr>
<td>G.2</td>
<td>70</td>
</tr>
<tr>
<td>Appendix H</td>
<td>75</td>
</tr>
<tr>
<td>H.1</td>
<td>75</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A list of electronic components.</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>A list of hardware and nonelectrical parts.</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>Simple pendulum measurement data.</td>
<td>23</td>
</tr>
<tr>
<td>4</td>
<td>Measured period and natural frequency of simple pendula with stationary pivots.</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>Introductory level results for simple pendula experiments.</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>Long, 20 g, simple pendulum experimental data.</td>
<td>27</td>
</tr>
<tr>
<td>7</td>
<td>Data collected for a 20 g, 16.5 cm simple pendulum.</td>
<td>27</td>
</tr>
<tr>
<td>8</td>
<td>Second data set collected for a 20 g, 16.5 cm simple pendulum.</td>
<td>28</td>
</tr>
<tr>
<td>9</td>
<td>Long, 100 g, simple pendulum experimental data.</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>Physical pendulum measurement data.</td>
<td>32</td>
</tr>
<tr>
<td>11</td>
<td>Measured and calculated periods of physical pendula with stationary pivots.</td>
<td>32</td>
</tr>
<tr>
<td>12</td>
<td>Long physical pendulum experimental data.</td>
<td>33</td>
</tr>
<tr>
<td>13</td>
<td>Short physical pendulum experimental data.</td>
<td>34</td>
</tr>
<tr>
<td>14</td>
<td>Results for physical pendula.</td>
<td>34</td>
</tr>
<tr>
<td>15</td>
<td>Cart frequency versus MCU output voltage data.</td>
<td>52</td>
</tr>
</tbody>
</table>
List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A simple pendulum with an oscillating pivot.</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>A physical pendulum with an oscillating pivot point.</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Illustration of the <em>common anode</em> connection.</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>A fully rendered perspective view of the <em>pendulum arm</em>.</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>A fully rendered <em>square bracket</em>.</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>Mass balancing; Determining the center of mass.</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>An illustration of a short physical pendulum.</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>Illustration of the motor–cart connection.</td>
<td>17</td>
</tr>
<tr>
<td>9</td>
<td>Illustration of the motor–cart connection geometry.</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>UV LED amplitude diagram.</td>
<td>20</td>
</tr>
<tr>
<td>11</td>
<td>Amplitude as a function of cart frequency plot for a simple pendulum.</td>
<td>29</td>
</tr>
<tr>
<td>12</td>
<td>Amplitude versus frequency plot of the 16.5 cm simple pendulum.</td>
<td>30</td>
</tr>
<tr>
<td>13</td>
<td>Amplitude versus frequency plot of the 100 g simple pendulum.</td>
<td>30</td>
</tr>
<tr>
<td>14</td>
<td>Amplitude evolution plot for a simple pendulum.</td>
<td>31</td>
</tr>
<tr>
<td>15</td>
<td>Amplitude as a function of cart frequency for a long physical pendula.</td>
<td>35</td>
</tr>
<tr>
<td>16</td>
<td>Amplitude as a function of cart frequency for a short physical pendula.</td>
<td>35</td>
</tr>
<tr>
<td>17</td>
<td>Plot of % difference versus amplitude angle for the small angle approximation.</td>
<td>47</td>
</tr>
<tr>
<td>18</td>
<td>Plot of the cart frequency versus the output voltage of the MCU.</td>
<td>52</td>
</tr>
<tr>
<td>19</td>
<td>A photograph of the entire apparatus.</td>
<td>57</td>
</tr>
<tr>
<td>20</td>
<td>A photograph of the 3D-printed components.</td>
<td>58</td>
</tr>
<tr>
<td>21</td>
<td>A photograph of a simple pendulum attached to the apparatus.</td>
<td>59</td>
</tr>
<tr>
<td>22</td>
<td>Photographs of the physical pendula.</td>
<td>60</td>
</tr>
<tr>
<td>23</td>
<td>A photograph of the MSD and MCU.</td>
<td>61</td>
</tr>
<tr>
<td>24</td>
<td>Plot of the position of a damped simple pendulum with a stationary pivot.</td>
<td>62</td>
</tr>
<tr>
<td>25</td>
<td>Plot of the position of a damped pendulum with fit data.</td>
<td>63</td>
</tr>
</tbody>
</table>
1 Introduction

Resonance is a mechanical and electromagnetic phenomenon. All bodies that undergo periodic motion can resonate when subjected to energy input at a specific frequency or frequencies. These specific frequencies, known as resonant frequencies, depend on the characteristics and boundary conditions of the body. If a bound system is driven at its resonant frequency, constructive superposition results in an increase of the amplitude of oscillations. For this reason, a resonating body may appear, to a naïve observer, to defy the principle of conservation of energy. Resonance can occur in, but is not limited to: bridges, electric circuits, musical instruments, and pendula.

The pendulum has been studied since Galileo Galilei published a manuscript in 1632 describing his observations about fundamental period relationships for a simple pendulum [1]. His work with the pendulum started the development of pendulum and pendular resonance experiments that continues to this day. One example of this development is a simple driven pendulum experiment known as a Barton pendulum, but does not allow for fine control over the drive frequency [2].

Commercial equipment is available for students to observe driven resonances in instructional physics labs today. In general, the equipment is complicated and expensive. Some of the most common driven pendulum experiments utilize magnetic phenomena to drive a pendulum [3, 4, 5]. There are also many simulations published about the driven pendulum [6, 7, 8]. Simulations are useful pedagogical tools, but leave the students without any real-world observations of resonance.

Other common driven pendulum experiments utilize torsion forces to drive the pendulum [9, 10]. Applying periodic torsion to a pendulum will drive its motion, but both linear and rotational damping must be considered [11]. Additionally, torsion pendula are usually quite complicated and expensive [12].

Although others have developed the theoretical description of a horizontally driven pen-
dulum [13, 14], there are few simple and affordable labs that have been developed to observe this system. There are no cart-driven pendulum labs to observe driven pendulum resonance using PASCO or Vernier dynamics carts. PASCO has a damped, driven oscillations experiment that uses Capstone software and rotary sensors to monitor and plot the amplitude as a function of drive frequency for an oscillating aluminum disk, but the apparatus is complicated and expensive [15]. All cart-driven pendulum experiments involve inverted physical pendula [16]. Each inverted pendulum apparatus that I found was technically complicated and expensive [10, 17, 18].

The main purpose of this project was to design, construct, and test an apparatus to observe the resonance behavior of mechanically driven pendula. Our goals for the design were that the apparatus be affordable, easily integrated into any physics laboratory, and produce consistent results. We assumed that most physics labs own PASCO or Vernier dynamics carts, and so we designed the apparatus to fit onto one of these cart designs. I contacted PASCO and confirmed that they do not have any pendulum attachments for their dynamics carts [19].

This document describes the design, construction and testing of a new pendulum attachment for common dynamics carts. We developed an experiment with the dynamics cart pendulum to help students observe and understand the phenomenon of driven resonance for simple and physical pendula.

I discuss the relevant theory pertaining to simple and physical pendula in Section 2. I describe the design and construction of the electronic, printed and pendulum components in Sections 3.1–3.3 and additional amplitude measuring techniques in Section 3.4. I give a detailed procedure of a possible experiment in Section 4.1 and show all of the data, results and calculations in Sections 4.2 and 4.3. I describe the conclusions about the apparatus and experiment as well as suggestions for future work, in Section 5.
2 Theory

This section covers the theoretical basis of the experiment described in this thesis. I discuss the simple and physical pendulum in Sections 2.1 and 2.2, respectively.

2.1 Simple Pendulum

A simple pendulum is a bob of mass $m$ suspended by a light, inextensible thread or string of length $l$. Introductory mechanics textbooks show that the period and natural frequency of a simple pendulum with a stationary pivot and undergoing small angular displacements are given by

$$T_o = 2\pi \sqrt{\frac{l}{g}}$$

and

$$\omega_o = \sqrt{\frac{g}{l}},$$

where $g$ is the acceleration due to gravity at the surface of the Earth [21]. I give further explanation of the small angle approximation in Appendix A.

A simple pendulum with an oscillating pivot point is illustrated in Fig. 1. During the experiment described in Section 4, the pendulum hangs from a pivot that oscillates along a horizontal line at angular frequency $\omega_D$. The horizontal position of the pivot is given by

$$X(t) = A_p \cos \omega_D t.$$  

We obtain the equation of motion of the simple pendulum-dynamics cart system in the absence of damping by applying the Lagrangian formalism. We choose the origin to be the equilibrium position of the pivot point. The center of mass of the bob has position components in the $x$ and $y$ directions given by

$$x(\theta,t) = l \sin \theta + A_p \cos \omega_D t$$

and

$$y(\theta) = -l \cos \theta.$$
Differentiating Eqs. (3) and (4) with respect to time yields the velocity components of the pendulum bob. The kinetic energy function, \( T(\theta, \dot{\theta}, t) \) depends on these velocity components. The potential energy function is \( V(\theta) = mgl(1 - \cos \theta) \) where we have chosen \( V = 0 \) when the pendulum bob hangs straight down (\( \theta = 0 \)). The Lagrangian function is

\[
L(\theta, \dot{\theta}, t) = T(\theta, \dot{\theta}, t) - V(\theta).
\] (5)

Substituting \( T(\theta, \dot{\theta}, t) \) and \( V(\theta) \) into Eq. (5) yields the Lagrangian

\[
L(\theta, \dot{\theta}, t) = \frac{ml^2}{2} \left[ \dot{\theta}^2 - \left( \frac{A_p \omega_D}{l} \right)^2 \dot{\theta} \cos \theta \sin \omega_D t + \left( \frac{A_p \omega_D}{l} \right)^2 \sin^2 \omega_D t - \frac{2g}{l} (1 - \cos \theta) \right].
\] (6)

Fig. 1: A simple pendulum with an oscillating pivot. The pendulum has a bob of mass \( m \), length \( l \), angular displacement \( \theta \) and an oscillating pivot. The pivot oscillates with angular frequency \( \omega_D \) and with a linear displacement amplitude of \( A_p \). The center of mass of the pendulum bob is labeled CM and marked with a small cross.
Note that the Lagrangian of the pendulum-cart system is time dependent. This indicates that the system is nonconservative [22]. The equation of motion of the pendulum bob is obtained from the Lagrangian by evaluating

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0. \quad (7)$$

Substituting Eq. (6) into Eq. (7) yields

$$\ddot{\theta} - \frac{A_p}{l} \omega_D^2 \cos \theta \cos \omega_D t + \frac{g}{l} \sin \theta = 0. \quad (8)$$

A detailed derivation of Eq. (8) is shown in Appendix B. We note that Eq. (8) simplifies to the equation of motion for a simple pendulum in the limiting cases when the drive frequency is zero or when the oscillation amplitude $A_p$ is zero.

During the experiments, the motion of each pendulum was damped. To account for the friction and air resistance experienced by the pendulum, we add an ad hoc damping term of $\beta \dot{\theta}$ to the equation of motion:

$$\ddot{\theta} - \frac{A_p}{l} \omega_D^2 \cos \theta \cos \omega_D t + \frac{g}{l} \sin \theta + \beta \dot{\theta} = 0. \quad (9)$$

I used a Python program to numerically solve Eq. (9) and calculate the maximum value of $\theta$ as a function of cart frequency, thereby generating the resonance curves shown in section 4.2.1.

### 2.2 Physical Pendulum

A physical or compound pendulum is a rigid body that swings under its own weight about a fixed axis of rotation. A physical pendulum with an oscillating pivot is illustrated in Fig. 2.

As for the simple pendulum, most physics textbooks derive the period and natural angular frequency for a physical pendulum when the angular displacement $\theta$ is much less than 1 rad ($\theta \ll 1$ rad).
Fig. 2: A physical pendulum with an oscillating pivot point. The pendulum has an oscillating pivot; the distance from the axis of rotation to the center of mass is denoted $l_{CM}$; $l$ denotes the distance from the pivot point to the bottom of the pendulum and $l'$ denotes the distance from the pivot point to the top of the pendulum; the pendulum swings with an angular displacement of $\theta$.

The period and natural frequency are given by [21]

$$T_o = 2\pi \sqrt{\frac{\sum I}{Mg l_{CM}}}$$

(10)

and

$$\omega_o = \sqrt{\frac{Mg l_{CM}}{\sum I}},$$

(11)

where $M$ represents the total mass, $\sum I$ represents the total rotational inertia about the pivot of the pendulum, and $l_{CM}$ represents the distance from the pivot to the center of mass of the physical pendulum.

As before, we apply the Lagrangian approach to obtain the equation of motion of the pendulum. The origin and pivot point motion are defined as was done for the simple pendulum. The position of the axis of rotation is given by $X(t)$ and the position of the center of mass
(CM) of the pendulum is given by

\[ x_{CM}(\theta, t) = l_{CM} \sin \theta + A_p \cos \omega_D t \]  

(12)

and

\[ y_{CM}(\theta) = -l_{CM} \cos \theta. \]  

(13)

The total kinetic energy, \( T(\theta, \dot{\theta}, t) \), will be the sum of the rotational kinetic energy and the translational kinetic energy of the pendulum. Differentiating Eqs. (12) and (13) with respect to time, substituting the resulting velocity functions into the kinetic energy equation, and subtracting the potential energy function yields the Lagrangian. The Lagrangian for the physical pendulum-cart system is

\[ L(\theta, \dot{\theta}, t) = \left[ \sum I_{CM} + Ml^2_{CM} \right] \ddot{\theta}^2 + \left[ \frac{MA_p \omega_D \sin \omega_D t}{2} \right] \left( A_p \omega_D \sin \omega_D t - 2l_{CM} \dot{\theta} \cos \theta \right) - Mgl_{CM}(1 - \cos \theta). \]  

(14)

We substituted this Lagrangian into Eq. (7) to obtain the equation of motion for an undamped physical pendulum with an oscillating pivot. The result is

\[ \left[ \sum I_{CM} + Ml^2_{CM} \right] \ddot{\theta} - MA_p \omega_D^2 l_{CM} \cos \theta \cos \omega_D t + Mgl_{CM} \sin \theta = 0. \]  

(15)

As for the simple pendulum, we added an ad hoc linear damping term to Eq. (15). The final result is

\[ \left[ \sum I_{CM} + Ml^2_{CM} \right] \ddot{\theta} - MA_p \omega_D^2 l_{CM} \cos \theta \cos \omega_D t + Mgl_{CM} \sin \theta + \beta \dot{\theta} = 0, \]  

(16)

which was numerically solved to calculate the resonance curves displayed in Section 4.2.2.
3 Design and Construction

I developed the apparatus for this experiment in three stages. The first stage was determining the necessary electronics required to operate and control the stepper motor. The next stage was the actual design and 3D-printing of the pendulum mount. The last stage was determining the necessary hardware and design for the final construction of the apparatus. Two important design criteria are that the apparatus is affordable and easily implemented into any undergraduate or graduate program. My design satisfied both design criteria.

3.1 Electronic Components

The electronic components in this experiment control the stepper motor, which drives the oscillatory motion of the dynamics cart. In general it will be necessary to be able to start/stop the motor at any time as well as have control of the speed of the motor. It takes two processes to properly control the stepper motor. The first process is creating and managing the input signal that directly controls the motor. A stepper motor operates in terms of micro-steps where the number of micro-steps for one full rotation of the motor shaft is dependent on the specifications of the stepper motor. We chose a Wantai 809 motor with step size of $0.9 \text{ deg}_\text{step}$ [23], so dividing $360 \text{ deg}_\text{rev}$ by the step size yields $400 \text{ steps}_\text{rev}$. The motor is rated for a maximum current of 1.7 A and has a voltage requirement of 12 V. Once the current, voltage and step size of the motor were determined, a micro-step driver can be chosen to satisfy these parameters. The micro-step driver connects directly to a power supply and the motor; this component operates the motor but requires a signal to be functional. The micro-step driver is operated by a pulse wave modulator (PWM) input signal. The PWM input signal can be produced in several ways. The simplest and most cost-effective way to create the PWM signal is with a micro-controller unit (MCU). The correct choice of MCU enables the motor to be started and stopped with the press of a button; the input frequency and the speed of the motor can be directly adjusted using the MCU. I found that there is a direct relationship between the frequency of the spin-shaft of the motor and the cart frequency, as shown in detail in Appendix C.
Table 1: A list of electronic components.

<table>
<thead>
<tr>
<th>Component</th>
<th>ID/Part #</th>
<th>Operating Voltage and Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Stepper Motor</td>
<td>Wantai 809</td>
<td>3 V and 1.7 A (max)</td>
</tr>
<tr>
<td>2. Micro-step Driver (MSD)</td>
<td>TB6600</td>
<td>9 - 42 V and 4 A (max)</td>
</tr>
<tr>
<td>3. Micro-controller (MCU)</td>
<td>Walfrontp1qv90aof</td>
<td>12 - 160 V</td>
</tr>
<tr>
<td>4. Power Supply</td>
<td>Chanzon</td>
<td>12 V and 3 A (min)</td>
</tr>
<tr>
<td>5. UV LED</td>
<td>ED YT05 U</td>
<td>3 - 3.3 V and 0.2 A (min)</td>
</tr>
<tr>
<td>6. Photo-luminescent Paper</td>
<td>Vvivid Glow Vinyl</td>
<td></td>
</tr>
<tr>
<td>7. Small Battery</td>
<td>Duracell</td>
<td>2 x 1.5 V or 1 x 3 V</td>
</tr>
</tbody>
</table>

Note: The total cost is $78 and the components are used to construct and control the motor.

A second way to create the PWM signal is to use an Arduino connected to the micro-step driver. An Arduino can be programmed to yield even more control over the circuit, allowing for the integration of additional monitoring equipment, making for an excellent addition to the set-up for upper level physics students.

Once the micro-step driver and MCU were chosen to match the specifications of the stepper motor, they were connected to run the motor. For a comprehensive list of the electronics we used in this experiment, see Table 1. Figure 3 illustrates the connections between the MCU and micro-step driver. This type of connection is known as a common anode connection [24], permitting complete control of the motor. It is important to note that the MCU used in this experiment has three PWM modes: high, 5.8 kHz to 127 kHz; medium, 590 Hz to 127 kHz; and low, 82 Hz to 2.3 kHz. The “low” setting should be used for this experiment because the range of possible simple pendulum lengths that fit on the pendulum arm correspond to the range of frequencies from approximately 1 Hz to 35 Hz.
Fig. 3: Illustration of the common anode connection. The connection between the MCU and the micro-step driver. The frequency of the motor is controlled by the potentiometer. There are two red buttons: one starts and stops the motor and the other reverses the direction of motor rotation.

3.2 Printed Components

PASCO and Vernier are the leading companies that supply dynamics carts to instructional physics labs. Neither company sells equipment to allow a pendulum to ride on the dynamics carts [25, 26]. We therefore designed a pendulum arm assembly consisting of two pieces to attach a pendulum to a dynamics cart. We used OpenSCAD [27] to design the pieces, which we then produced with a 3D printer. We tried using as little material as possible while keeping enough structural integrity to support the pendulum. The OpenSCAD code we wrote is shown in Appendix D.

3.2.1 Pendulum Arm

The pendulum arm has a pivot point about 300 mm above the top of the cart. This allows for several pendulum lengths to be tested within the experiment. The pendulum arm is illustrated in Fig. 4 and takes between five and ten hours to print. The shaft to which a pendulum is attached has a slit for the string supporting the pendulum bob, enabling easy control of the
length of a simple pendulum. The shaft has a diameter of a quarter inch (6.35 mm) so that the inner race of the bearing will fit snugly, anchoring the physical pendulum with no clips. The pendulum arm is essentially a vertical cantilever that is stiffened by long, narrow gussets connecting the individual arms of the pendulum arm to the base.

### 3.2.2 Square Bracket

The square bracket attaches the pendulum arm to the dynamics cart and is shown in Fig. 5. The bracket has two through holes that are intentionally positioned to match threaded holes in PASCO dynamics carts. The base of the pendulum arm fits under the middle of the bracket.
The two slits cut on the narrow part of the bracket allow the gussets of the pendulum arm to pass through the bracket. The through holes are for 10–32 bolts (approximately 5 mm in diameter). The bracket takes two and a half to four hours to print.

Fig. 5: A fully rendered *square bracket*. Views of the square bracket are (a) a perspective view of the bracket and (b) a top view of the bracket.
3.3 Pendulum Design and Hardware

I describe the design of the pendula and the mechanism that connects the cart to the motor in this section. All of the necessary parts are listed in Table 2 and were chosen because they are inexpensive, dependable and durable.

Table 2: A list of hardware and nonelectrical parts.

<table>
<thead>
<tr>
<th>Component</th>
<th>ID/Part #</th>
<th>Use/Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. Dynamics Track</td>
<td>Al-track, 1.2 m (ME-9493)</td>
<td>Track: maintains cart direction.</td>
</tr>
<tr>
<td>10. Two Cart screws</td>
<td>10-32 3/4”</td>
<td>Screws for square bracket.</td>
</tr>
<tr>
<td>11. Universal Mounting Hub</td>
<td>Pololu 5 mm (1083)</td>
<td>Connection to motor’s spin-shaft.</td>
</tr>
<tr>
<td>12. Threaded rod</td>
<td>Sullivan 4-40</td>
<td>Connects hub and cart.</td>
</tr>
<tr>
<td>13. Swivel Ball Links</td>
<td>Du-Bro 861</td>
<td>Connect to end of threaded rod.</td>
</tr>
<tr>
<td>15. Two Aluminum Beams</td>
<td>servocity (585420)</td>
<td>7.7” arm for pendulum.</td>
</tr>
<tr>
<td>16. Motor Mount</td>
<td>NEMA 17</td>
<td>Mount motor to table or track.</td>
</tr>
<tr>
<td>17. Motor Damper</td>
<td>NEMA 17</td>
<td>Damp the vibrations of the motor.</td>
</tr>
</tbody>
</table>

Note: The total cost is $68.78 without the cart or track and $253.78 with the cart and track.

3.3.1 Simple Pendulum Design

The simple pendulum is formed using the pendulum arm shown in Fig. 4. A light, “in-extensible” thread sits in the vertical slit and is then wound around the shaft. Threading the string through the slit allows the string to hang vertically. A picture of the string and vertical slit is shown in Appendix E. The distance from the bottom of the slit to the center of mass of the pendulum bob is the effective length of the pendulum. I used pendulum bobs that ranged in mass from 20 - 100 g. The simple pendulum resembled the diagram in Fig. 1. The suggested bob for this experiment is one from a typical hanging mass set. Physics labs usually already own mass sets and the bob will be cylindrical instead of spherical; cylinders will experience slightly less air resistance [28] and their centers of mass are easier to determine.

Determining the center of mass of the bob and making a measurement from the pivot point to the center of mass is an excellent exercise for students. Using the string that will hold the
pendulum bob, students can balance the hanging mass horizontally with the thread, to determine the location of the center of mass. The balancing technique is displayed in Fig. 6.

![Image of pendulum](image)

Fig. 6: A 20 g mass is pictured, balanced horizontally from a loop in the string on the pendulum arm. The center of mass is located at the center of the circular cross section formed by the loop that balances the mass.

3.3.2 Physical Pendulum Design

The physical pendulum is illustrated in Fig. 7 and photographs of the different physical pendula can be found in Appendix E. The fundamental components of the physical pendulum are the dual ball bearing hub and aluminum beam 1, numbers 14 and 15 in Table 2. The bearing fits snugly onto the shaft at the top of the pendulum arm. The bearing acts as a hollow annulus and was chosen such that distance between the holes in the bearing equals the distance between holes on the aluminum beam. The beam is bolted to the bearing and acts as the arm of the physical pendulum. The holes along the length of the aluminum beam allow for easy mass attachment to change the value of $l_{CM}$ and $\sum I_{CM}$.

Before mounting the physical pendulum, the center of mass, the total mass and the moment of inertia about the axis of rotation must be determined. The center of mass of a physical pendulum was determined using the same balancing technique that was shown in Fig. 6. The mass of each piece of the physical pendulum were measured; the distance from center of the pivot to the center of mass of each piece were measured. For simplicity, the hub was treated as a hollow annular cylinder and the holes cut into the hub were ignored. Each bolt was treated as a point mass and the beam was approximated as a metal rod, again, ignoring the holes in the beam. All of the bodies were assumed to have uniform densities and rotate about the central
axis of the pivot. The necessary equations to determine the moment of inertia are given below.

\[
I_{\text{cyl}} = \frac{m_{\text{hub}}}{2} \left[ r_{\text{outer}}^2 + r_{\text{inner}}^2 \right].
\]  \hspace{1cm} (17)

The moment of inertia for the bolts was assumed to be that of a point mass and is given by

\[
I_{\text{bolt}} = m_{\text{bolt}} r_{\text{bolt}}^2.
\]  \hspace{1cm} (18)

I assume that the aluminum beam is a solid rod, rotating about the pivot point, which is \( l' \) from the top of the beam.

\[
I_{\text{Al,beam}} = \frac{m_{\text{beam}}}{3(l + l')} \left[ r^3 + l'^3 \right].
\]  \hspace{1cm} (19)

I use the parallel-axis theorem to help derive the moment of inertia for the physical pendula, which is given by

\[
I = I_o + md^2.
\]  \hspace{1cm} (20)
where \( I_o \) is defined as the moment of inertia of a component of the physical pendulum rotating about the pivot and \( d \) is defined as the perpendicular distance between the center of mass and the center of mass of the component.

### 3.3.3 Connecting the Cart and Motor

One intriguing aspect of resonance is how little the energy input per oscillation cycle can be to obtain large amplitude oscillations, as long as the driving frequency matches the resonant frequency of the pendulum. Working through this experiment will allow students to have greater appreciation of this. I designed the motor-to-cart connection to ensure that the cart oscillation amplitude is small and to therefore show that very little energy input per cycle can cause large oscillations.

The universal mounting hub (Item 11 in Table 2) was connected to the spin-shaft of the motor. It is a small cylinder that has evenly spaced, threaded holes for attaching links to other objects. The center of the cylinder has a non-threaded hole that is 5 mm in diameter (to match the diameter of the spin-shaft) and is equipped with a set screw so it can be secured to the spin-shaft.

The connection between the hub and the cart must be rigid and inextensible so the motion is periodic. This rigid connection consists of a thin threaded rod with swivel ball links attached to each end (Items 12 and 13 in Table 2). One of the ball links is secured to the universal mounting hub while the other is secured to the cart. The swivel ball links allow for the hub to rotate one end of the threaded rod in complete circles while allowing the orientation of the rod to freely change. This forces the cart to move forth and back, whenever the rod pushes or pulls the cart. When the hub rotates the rod in any direction that is not parallel to the motion of the cart, the rod will change its orientation but will not change the direction of the cart. The connection mechanism for the cart and the motor is illustrated in Fig. 8.

We had to consider the motion of the rod when we define the horizontal position of the pivot. The geometry for the horizontal position of the cart is illustrated in Fig. 9 and I used this geometry to define the actual horizontal position of the cart and, therefore, the pivot.
Fig. 8: Illustration of the motor–cart connection. A top view (above) and a side view (below) of the hardware and connection between the motor and the cart. Item 1 is the mounting hub, item 2 is the cart, item 3 is the threaded rod, items 4 are the swivel links, item 5 is the spin-shaft and item 6 is the motor. Item 7 is a locknut, that anchors the bolt that connects the swivel link to the mounting hub.

Fig. 9: Illustration of the motor–cart connection geometry. A top view illustration of the three geometric lengths we used to determine the horizontal position of the cart as a function of time.

The position of the pivot is given by

\[ X(t) = r \sin \omega_D t + d \sqrt{1 - \left(\frac{r}{d}\right)^2 \cos^2 \omega_D t}, \]

where \( d \) is the length of the threaded connecting rod and \( r \) is the radius of the circular motion that the left side of the threaded rod undergoes. The position of the cart will actually depend
on two terms, instead of the single sine function used in Section 2. The reason that we ignored the extra term in the $X(t)$ is because of the sizes of $r$ and $d$ in the experiment. The apparatus I used during the experiments has an $r = 6.5 \text{ mm}$ and $d = 33.6 \text{ cm}$. When I substitute the actual values of $r$ and $d$ into Eq. 21, the term under the square root has a minimum value of 0.9998, so to a good approximation, the equation becomes

$$X(t) = r \sin \omega Dt + d. \quad (22)$$

Adding a constant to the position of the pivot in Section 2 would not have affected the Lagrangian formalism used to derive the equation of motion. We therefore ignored the additional term in Eqs. (21) and (22).

### 3.4 Measuring Angular Displacement Amplitude

The experiment depends on measuring the angular displacement amplitude of each pendulum. We expect maximum oscillation amplitudes when the pendulum stand oscillates at the resonant frequency of a pendulum. An indicator that the oscillation frequency matches the resonant frequency of the pendulum is that the pendulum moves forth and back at the maximum amplitude without coming to rest at its initial position. In other words, the pendulum will oscillate in a stable normal mode, instead of some superposition of modes that could yield a beating oscillation. Plotting the measured amplitude versus the driving frequency produces a relationship called a resonance curve that is expected to have an obvious maximum at the resonant frequency of the pendulum. Several ways that students can measure the amplitude of the pendulum are described in this section.

#### 3.4.1 Visual Measurements

I found that the most time efficient amplitude measuring technique is to visually measure the angular displacement of the pendulum. It is easy for students to attach a protractor onto the pendulum arm assembly to measure the angular displacement of each pendulum. It was important to align my line-of-sight directly with one of the maximum amplitude locations of
the cart to minimize any parallax effects. The major disadvantage to visually measuring the amplitude is the large uncertainty in the measurements. Uncertainties in visual measurements, for most people, should fall between one and three degrees.

3.4.2 Camera Phones

In the 21st century, it is likely that most of the students in a physics lab have cellular telephones with high-quality cameras that record video. Some labs are even incorporating cellular telephones as data collection devices [29]. Once the pendulum is in motion, students can record video of the pendulum when it reaches its maximum amplitude on their phones. An advantage of video is that students can pause the footage when the pendulum is at maximum amplitude and read the angular displacement from the string and protractor in the video. A disadvantage with camera phones is the lack of control of exposure time. Large exposure time can result in motion blur and lead to greater uncertainties. With the angular displacement and the measured length of the pendulum, the amplitude can be determined. One should align the camera directly with the pendulum when recording to avoid parallax issues.

3.4.3 UV LED

Utilizing a phosphorescent background screen and a UV LED attached to a pendulum bob (or at the center of mass of the physical pendulum) can trace the path of a pendulum. As a pendulum swings, the LED will leave a glowing trail on the screen behind the pendulum, illustrated in Fig. 10. Once the trail is on the screen, students can use a protractor to measure the maximum angular displacement of the pendulum. With the measured angular displacement and the length of the pendulum, the amplitude can be determined.
4 Experiment

The purpose of the experiment is to observe pendula undergoing driven resonance both qualitatively and quantitatively. The apparatus of the experiment was designed so the driving frequency of the cart and the amplitude of the pendulum swing are measured values. Amplitude measurement techniques are discussed in Section 3.4. During this experiment, I taped a protractor to the pendulum arm so the amplitude of the pendulum could be measured visually or with a camera (discussed in Section 3.3.2). This experiment yields qualitatively accurate results despite the size of the uncertainties in visually measured amplitudes. The output frequency of the MCU can be monitored with an oscilloscope or sensitive voltmeter. The relationship between the output signal of the MCU and the cart frequency was experimentally determined and is shown in Appendix C.

It is important to determine the accuracy and reliability of any student experiment. I determined the experimental values for the resonant frequencies of each pendulum by measuring the amplitude as a function of frequency and then identifying the frequency that maximized the
amplitude. These frequencies are compared to values calculated using Eqs. (1) and (2) as well as simulated amplitude data from solving Eqs. 9 and 16. Another way for students to estimate the natural frequency is to measure the period of each pendulum when it experiences small angular displacements (see Appendix A), with a stationary pivot. This was done by timing ten oscillations for each pendulum; I used small angular displacements for each pendulum, except the short physical pendulum. The frequency of small oscillations was compared to the resonant frequency determined by the measured amplitude as a function of cart frequency, when the pivot was oscillating. I compare these values, even though the oscillator is really a non-linear oscillator, because it is the level that an introductory lab will test accuracy. For upper level laboratories, the predicted resonant frequencies should be determined by numerically integrating the equations of motion for each pendulum with an oscillating pivot. The procedure for data collection is described in Section 4.1. I display all of the data, measurement uncertainties, and results in Section 4.2. I outline sample calculations in Section 4.3.

4.1 Procedure

Prior to making any frequency measurements, it was necessary to determine the masses of all the pieces that comprise each pendulum. The mass should be measured using a triple beam balance or an electronic scale. The center of mass of each simple pendulum bob and of each physical pendulum was determined by creating a loop in a thread and affixing the other end to the pendulum arm. Each simple pendulum bob and physical pendulum were threaded through the loop at the bottom of the string such that it was suspended horizontally. The balancing technique is displayed in Fig. 7. Keeping the string in place, a line was traced onto the simple pendulum bob or physical pendulum marking the center of mass. Finally the length of each pendulum was measured from the determined center of mass to the pivot. I display measurement data in Table 3.

With all the necessary measurements completed, I attached a pendulum to the pendulum arm. Before turning on the motor, the period of the undriven pendulum was experimentally de-
terminated. The pendulum was set into motion with small oscillations and 5–10 complete cycles were timed with a stopwatch. From this the period for one oscillation was determined; inverting the period yields an estimate for the resonant frequency for the pendulum. I display the data for simple pendula in Table 4.

The motor was turned on and set to a frequency of about 0.5 Hz. Five to ten oscillations of the cart were timed and the maximum amplitude of the pendulum was observed and recorded. The frequency of the cart was then increased by adjusting the potentiometer. Because the potentiometer is very sensitive, it was adjusted by only a small amount. The change in frequency was visually imperceptible, if done correctly. Interestingly enough, it was possible to hear the slight changes in frequency of the motor. Once adjusted, the measurements were repeated. Five or six measurements were performed for drive frequencies on either side of the resonant frequency. Finally the maximum amplitude versus drive frequency data were plotted. All collected experimental data and uncertainties are displayed in Tables 5, 6, and 7. Qualitative observations of resonance were made throughout the experiment.

4.2 Data and Results

I display the data and results from the experiments with the simple and physical pendula in Sections 4.2.1 and 4.2.2, respectively. We produce plots of maximum amplitude versus drive frequency with data from these experiments. We expect the maximum amplitude versus drive frequency plots to be the same shape, no matter the type of pendulum we are testing. We expect the plot to show an asymmetric peak with a steep slope for frequencies below the peak and that the curve will fall less rapidly for frequencies above the peak. In this experiment, all of the pendula we tested were, at least, slightly damped. We expect increasing damping to broaden the resonance peak. We determined the damping constants experimentally and give an example analysis in Appendix F. The damping allows the pendulum to oscillate in less steady, anharmonic modes, especially when the cart frequency is near, but not equivalent to, the resonant frequency of the pendulum. It is important to note that we could not treat the pendulum
as a simple harmonic oscillator because the observed amplitudes of oscillation were larger than 40° when the drive frequency was near the resonant frequency of the pendulum. Observations of the evolution of a pendulum position with respect to time are consistent with the idea that the pendulum is acting as a self-limiting oscillator with a resonant frequency slightly less than the natural frequency.

4.2.1 Simple Pendula Data and Results

I tested three different simple pendula; Table 3 contains all mass and length measurements for each simple pendulum. Table 4 contains the theoretical period and frequency I calculated for small angle oscillations and the experimental periods and frequencies that I collected when the pivot was not driven. The uncertainties in Table 3 are standard measurement uncertainties dependent on the measuring device that I used. In this case, I made measurements using a standard meter stick with uncertainty of ± 0.5 mm and a digital scale with uncertainty of ± 0.5 g. I describe the calculations of the uncertainties in the measured periods and frequencies shown in Table 4.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>Mass (g)</th>
<th>Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long simple</td>
<td>20 ± 0.5</td>
<td>26.5 ± 0.05</td>
</tr>
<tr>
<td>Short simple</td>
<td>20 ± 0.5</td>
<td>16.5 ± 0.05</td>
</tr>
<tr>
<td>Long simple</td>
<td>100 ± 0.5</td>
<td>26.5 ± 0.05</td>
</tr>
</tbody>
</table>

Note: The bob mass and length of each of the simple pendula.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>(T_{\text{meas}}) (s)</th>
<th>(T_{\text{calc}}) (s)</th>
<th>(f_{\text{meas}}) (Hz)</th>
<th>(f_{\text{calc}}) (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long (20 g)</td>
<td>1.03 ± 0.02</td>
<td>1.033 ± 0.001</td>
<td>0.969 ± 0.017</td>
<td>0.968 ± 0.003</td>
</tr>
<tr>
<td>Long (100 g)</td>
<td>1.04 ± 0.02</td>
<td>1.033 ± 0.001</td>
<td>0.960 ± 0.018</td>
<td>0.968 ± 0.003</td>
</tr>
<tr>
<td>Short (20 g)</td>
<td>0.82 ± 0.02</td>
<td>0.815 ± 0.0013</td>
<td>1.220 ± .030</td>
<td>1.227 ± 0.0037</td>
</tr>
</tbody>
</table>

Note: The initial angular displacement applied to each pendulum was 10°.
I used the calculated small-amplitude frequencies for each pendulum to scale the measured drive frequencies of our data. The data I collected for each simple pendulum are displayed in Tables 6–9. I calculated the percent uncertainties by dividing my average response time [20] by the measured times. I used the percent uncertainties in measured time to calculate the uncertainties in cart frequencies. I show an example of the uncertainty calculation for experimental frequencies in Section 4.3.

I plot maximum amplitude as a function of cart frequency for the simple pendula in Figs. 11, 12 and 13. I measured the maximum amplitudes visually while the cart was in motion yielding large uncertainties; the uncertainty was ±3°. Even with large uncertainties, it was clear to me when the drive frequency was close to the resonant frequency of the pendulum. The solid lines on the plots represent the theoretical maximum amplitude as a function of cart frequency. I determined the theoretical resonance curves by numerically integrating Eq. (9) with a Python program. I display an example of the Python program in Appendix G. Table 5 contains the experimentally determined resonant frequencies for each simple pendulum that I tested.

Table 5: Introductory level results for simple pendula experiments.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>$f_{\text{experimental}}$ (Hz)</th>
<th>$f_{\text{natural}}$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long (20 g)</td>
<td>0.971 ± 0.017</td>
<td>0.969 ± 0.019</td>
</tr>
<tr>
<td>Short (20 g)</td>
<td>1.111 ± 0.025</td>
<td>1.220 ± 0.030</td>
</tr>
<tr>
<td>Long (100 g)</td>
<td>0.966 ± 0.019</td>
<td>0.960 ± 0.018</td>
</tr>
</tbody>
</table>

Note: The $f_{\text{natural}}$ values are taken from timing the period of each pendulum with a stationary pivot, displayed in Table 4.

Table 5 is an example of the final results that an introductory lab may require of the students. Most introductory students will not be able to plot theoretical curves to compare with their data. The experimental small-oscillation frequency for the long, 20 g pendulum is determined to be $0.971 \pm 0.017$ Hz and the theoretical frequency was calculated to be $0.969 \pm 0.003$ Hz. The uncertainty ranges overlap, indicating that the textbook model for small oscillations is valid. As shown in Table 4, I obtained similar results for the 100 g long simple pendulum. The
two long pendula also had overlapping uncertainty ranges, supporting a mass independence of the resonant frequency of a simple pendulum.

We note that we cannot directly compare the measured resonant frequency of the pendulum to the small-oscillation frequency because the amplitude of the pendulum is not in the small angle regime for any drive frequencies within 10% of the predicted resonant frequency. This is illustrated in the resonance curves shown in Figs. 11–13. This explains why the peak of the measured resonance curves do not occur at $\frac{\omega_D}{\omega_o} = 1$. Nonetheless, introductory students can observe the resonance behavior, measure the resonance curve, and conclude that the resonance peak does not agree with the predicted, small-oscillation frequency.

As expected, the theoretical resonance curves for all of the simple pendula maintained the same shape. I determined the damping constants used in the simulations by tracking the motion of each pendulum with a stationary pivot and plotting the horizontal motion with respect to time. I then plotted a general solution to the equation of motion on the same axes and used an Excel minimization function to obtain best fit values for the parameters in the general solution. I note that an interesting aspect of the resonance curves show that the actual resonant frequency for the simple pendula occur at a lower frequency than the small-oscillation frequencies of each pendulum. The offset of the resonant peak for any of the simple pendula that I tested support the notion that the pendula are not actually acting as harmonic oscillators but instead as non-linear, or anharmonic oscillators. The pendulum with a driven pivot is a self-limiting oscillator, due to the sinusoidal dependence on the angular displacement in the equation of motion given by Eq. 9. The major support to this claim came observationally, as I measured maximum amplitudes during the experiment. I found it impossible, at any drive frequency, to observe a pendulum with a constant oscillation amplitude. I observed the pendula to undergo beating behavior, no matter what the drive frequency was. To model this, we wrote another Python program that predicts the position of the pendulum bob with respect to time, for a given drive frequency. We found that no matter what drive frequency we input into the simulation, the resulting motion of the pendulum would undergo
beats such that the oscillation amplitude would start from zero and rise continuously to its maximum amplitude, only to fall back continuously to zero again. I show an example of the amplitude evolution for a simple pendulum in Fig. 14, where we set the drive frequency to the natural frequency for the pendulum.

The idea that pendula act as self-limiting oscillators is supported theoretically by an equation that relates the amplitude of oscillation to the resonant frequency of a pendulum. The equation for the amplitude of a pendulum driven at a steady resonant frequency, with small damping, is derived in many introductory mechanics texts and is given by [21]

$$\omega \approx \omega_o \left(1 - \frac{A^2}{8}\right)^{1/2},$$

(23)

where $\omega$ is the angular frequency, $A$ is the amplitude measured in radians, and $\omega_o$ is the natural frequency. This equation shows that $\omega$ will change dependent on the amplitude of the oscillations. When the amplitudes of the oscillations increase, the natural frequency is multiplied by a smaller number and the new resonant frequency of the pendulum will be slightly smaller. Our apparatus does not adjust the drive frequency as a function of oscillation amplitude, so as the amplitude of the pendulum increases, the system will not be oscillating at the resonant frequency any longer and the amplitude of the oscillations will begin to decrease.

An addition to the apparatus could be made, by future experimenters, to account for the self-limiting behavior of the pendula. A rotational motion sensor could be installed at the pivot that would send amplitude information back to the MCU. If the amplitude gets too large, the MCU would adjust the output frequency so the pendulum would continue to resonate. A different MCU and expensive sensors would have to be implemented, increasing the total cost of the apparatus.
Table 6: Long, 20 g, simple pendulum experimental data.

<table>
<thead>
<tr>
<th>Time for 10 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.52</td>
<td>0.540 ± 0.006</td>
<td>3</td>
</tr>
<tr>
<td>16.96</td>
<td>0.590 ± 0.007</td>
<td>4</td>
</tr>
<tr>
<td>14.99</td>
<td>0.667 ± 0.008</td>
<td>4.5</td>
</tr>
<tr>
<td>13.6</td>
<td>0.735 ± 0.010</td>
<td>4</td>
</tr>
<tr>
<td>12.84</td>
<td>0.779 ± 0.011</td>
<td>4.5</td>
</tr>
<tr>
<td>11.9</td>
<td>0.840 ± 0.013</td>
<td>6</td>
</tr>
<tr>
<td>11.6</td>
<td>0.862 ± 0.014</td>
<td>12</td>
</tr>
<tr>
<td>10.3</td>
<td>0.971 ± 0.017</td>
<td>62</td>
</tr>
<tr>
<td>8.42</td>
<td>1.188 ± 0.023</td>
<td>12</td>
</tr>
<tr>
<td>8.02</td>
<td>1.247 ± 0.030</td>
<td>9.5</td>
</tr>
<tr>
<td>7.35</td>
<td>1.361 ± 0.041</td>
<td>6.5</td>
</tr>
<tr>
<td>6.63</td>
<td>1.508 ± 0.051</td>
<td>5</td>
</tr>
<tr>
<td>5.86</td>
<td>1.706 ± 0.062</td>
<td>4</td>
</tr>
<tr>
<td>5.5</td>
<td>1.818 ± 0.067</td>
<td>4</td>
</tr>
</tbody>
</table>

Note: The amplitudes were visually determined. Ten complete cart oscillations were timed for each frequency.

Table 7: Data collected for a 20 g, 16.5 cm simple pendulum.

<table>
<thead>
<tr>
<th>Time for 10 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>28.19</td>
<td>0.494 ± 0.004</td>
<td>4</td>
</tr>
<tr>
<td>22.95</td>
<td>0.552 ± 0.006</td>
<td>4.5</td>
</tr>
<tr>
<td>19.35</td>
<td>0.661 ± 0.007</td>
<td>5</td>
</tr>
<tr>
<td>16.74</td>
<td>0.689 ± 0.011</td>
<td>5</td>
</tr>
<tr>
<td>14.74</td>
<td>0.880 ± 0.013</td>
<td>7</td>
</tr>
<tr>
<td>11.90</td>
<td>0.988 ± 0.018</td>
<td>7.5</td>
</tr>
<tr>
<td>10.86</td>
<td>1.049 ± 0.021</td>
<td>12</td>
</tr>
<tr>
<td>9.98</td>
<td>1.136 ± 0.025</td>
<td>42</td>
</tr>
<tr>
<td>8.59</td>
<td>1.230 ± 0.030</td>
<td>61</td>
</tr>
<tr>
<td>8.03</td>
<td>1.297 ± 0.035</td>
<td>37</td>
</tr>
<tr>
<td>7.54</td>
<td>1.408 ± 0.044</td>
<td>20</td>
</tr>
<tr>
<td>6.72</td>
<td>1.669 ± 0.078</td>
<td>7</td>
</tr>
<tr>
<td>4.35</td>
<td>2.632 ± 0.121</td>
<td>12</td>
</tr>
</tbody>
</table>

Note: Ten complete cart oscillations were timed for each frequency.
Table 8: Second data set collected for a 20 g, 16.5 cm simple pendulum.

<table>
<thead>
<tr>
<th>Time for 5 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.90</td>
<td>1.282 ± 0.066</td>
<td>43</td>
</tr>
<tr>
<td>3.77</td>
<td>1.327 ± 0.070</td>
<td>23</td>
</tr>
<tr>
<td>3.54</td>
<td>1.412 ± 0.080</td>
<td>19</td>
</tr>
<tr>
<td>3.88</td>
<td>1.289 ± 0.066</td>
<td>30</td>
</tr>
<tr>
<td>4.045</td>
<td>1.236 ± 0.061</td>
<td>55</td>
</tr>
<tr>
<td>4.07</td>
<td>1.229 ± 0.060</td>
<td>50</td>
</tr>
<tr>
<td>3.83</td>
<td>1.305 ± 0.068</td>
<td>25</td>
</tr>
<tr>
<td>3.96</td>
<td>1.263 ± 0.064</td>
<td>41</td>
</tr>
<tr>
<td>4.05</td>
<td>1.235 ± 0.061</td>
<td>51</td>
</tr>
<tr>
<td>4.10</td>
<td>1.220 ± 0.059</td>
<td>59</td>
</tr>
<tr>
<td>4.16</td>
<td>1.202 ± 0.058</td>
<td>62</td>
</tr>
<tr>
<td>4.30</td>
<td>1.163 ± 0.054</td>
<td>85</td>
</tr>
<tr>
<td>4.50</td>
<td>1.111 ± 0.049</td>
<td>95</td>
</tr>
<tr>
<td>4.60</td>
<td>1.087 ± 0.047</td>
<td>37</td>
</tr>
<tr>
<td>4.70</td>
<td>1.064 ± 0.045</td>
<td>12</td>
</tr>
</tbody>
</table>

Note: During the data collection I focused on drive frequencies close to the resonant frequency of the pendulum. Five complete cart oscillations were timed for each frequency.

Table 9: Long, 100 g, simple pendulum experimental data.

<table>
<thead>
<tr>
<th>Time for 10 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.85</td>
<td>0.438 ± 0.004</td>
<td>3.5</td>
</tr>
<tr>
<td>21.42</td>
<td>0.467 ± 0.004</td>
<td>4</td>
</tr>
<tr>
<td>15.61</td>
<td>0.641 ± 0.008</td>
<td>4</td>
</tr>
<tr>
<td>13.66</td>
<td>0.732 ± 0.011</td>
<td>5</td>
</tr>
<tr>
<td>12.54</td>
<td>0.797 ± 0.013</td>
<td>12</td>
</tr>
<tr>
<td>11.32</td>
<td>0.883 ± 0.016</td>
<td>37</td>
</tr>
<tr>
<td>10.35</td>
<td>0.966 ± 0.019</td>
<td>65</td>
</tr>
<tr>
<td>9.98</td>
<td>1.002 ± 0.020</td>
<td>29</td>
</tr>
<tr>
<td>8.93</td>
<td>1.120 ± 0.025</td>
<td>15</td>
</tr>
<tr>
<td>7.81</td>
<td>1.280 ± 0.033</td>
<td>7</td>
</tr>
<tr>
<td>6.7</td>
<td>1.493 ± 0.045</td>
<td>14</td>
</tr>
<tr>
<td>5.56</td>
<td>1.799 ± 0.065</td>
<td>11</td>
</tr>
<tr>
<td>4.03</td>
<td>2.481 ± 0.123</td>
<td>10</td>
</tr>
</tbody>
</table>

Note: Ten complete cart oscillations were timed for each frequency.
Fig. 11: Amplitude as a function of cart frequency plot for a simple pendulum. Measured maximum amplitude of a 20 g simple pendulum as a function of pivot oscillation frequency data listed in Table 6. The experimental amplitude measurements are represented by the blue points. The theoretical amplitude curve is represented by the black line, which was numerically determined with a Python program. The code for the numerical integration is given in Appendix G.
Fig. 12: Amplitude versus frequency plot of the 16.5 cm simple pendulum. Plot of the 20 g simple pendulum data. The black curve represents the theoretical amplitude as a function of cart frequency curve.

Fig. 13: Amplitude versus frequency plot of the 100 g simple pendulum. Plot of the 26.5 cm simple pendulum data. The experimental data is represented with blue points. The black curve represents the theoretical amplitude as a function of cart frequency curve.
Fig. 14: Amplitude evolution plot for a simple pendulum. This plot predicts the evolution of the amplitude of a simple pendulum for a given drive frequency. We set the drive frequency equal to the predicted resonant frequency of the 26.5 cm, 20 g simple pendulum.

4.2.2 Physical Pendula Data and Results

Two different physical pendula were tested during this experiment. The long and short physical pendula were constructed of hardware listed in Table 10. The short physical pendulum is illustrated in Fig. 8. Table 11 contains the period and frequency of each pendulum as measured with a stationary pivot point and the period and frequency for the short pendulum as predicted using Eqs. (10) and (11), assuming small oscillations.

Comparing the measured and calculated small oscillation periods for the short physical pendulum shows that the uncertainties overlap. The measured values from Table 11 can be compared to the experimentally determined resonant frequencies of the physical pendula in introductory labs.

Tables 12 and 13 contain the data for the long and short physical pendula, respectively. The times for 5–10 complete oscillations are recorded to determine the cart frequency of each trial. Multiple oscillations were timed to reduce the uncertainty in frequency measurements.
Table 10: Physical pendulum measurement data.

<table>
<thead>
<tr>
<th>Components</th>
<th>Mass (g)</th>
<th>r&lt;sub&gt;CM&lt;/sub&gt; (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al Beam 1</td>
<td>28.6</td>
<td>l = 19, l' = 1.5</td>
</tr>
<tr>
<td>Al Beam 2</td>
<td>28.3</td>
<td>L = 20.5</td>
</tr>
<tr>
<td>Bearing Hub</td>
<td>11.4</td>
<td>r&lt;sub&gt;outer&lt;/sub&gt; = 1.27, r&lt;sub&gt;inner&lt;/sub&gt; = 0.3175</td>
</tr>
<tr>
<td>Bolt/Nut 1</td>
<td>2.3</td>
<td>7.6</td>
</tr>
<tr>
<td>Bolt/Nut 2</td>
<td>2.3</td>
<td>18.5</td>
</tr>
<tr>
<td>Bolt/Spacer 1</td>
<td>1.6</td>
<td>0.98</td>
</tr>
<tr>
<td>Bolt/Spacer 2</td>
<td>1.6</td>
<td>0.98</td>
</tr>
<tr>
<td>Short Physical Pendulum</td>
<td>43.3</td>
<td>l&lt;sub&gt;CM&lt;/sub&gt; = 5.9</td>
</tr>
<tr>
<td>Long Physical Pendulum</td>
<td>76.5</td>
<td>l&lt;sub&gt;CM&lt;/sub&gt; = 11.2</td>
</tr>
</tbody>
</table>

Note: The measurements of the components of each physical pendulum. The uncertainty in all of the length measurements is 1 mm and the uncertainty in all of the mass measurements is 0.05 g. The distance between each component and the pivot point is denoted as r<sub>CM</sub>. The variables l, l' and l<sub>CM</sub> are illustrated in Fig. 7.

Table 11: Measured and calculated periods of the physical pendula with stationary pivots.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>T&lt;sub&gt;meas&lt;/sub&gt; (s)</th>
<th>T&lt;sub&gt;calc&lt;/sub&gt; (s)</th>
<th>f&lt;sub&gt;meas&lt;/sub&gt; (Hz)</th>
<th>f&lt;sub&gt;calc&lt;/sub&gt; (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P. Short</td>
<td>0.72 ± 0.02</td>
<td>0.71 ± 0.03</td>
<td>1.40 ± 0.04</td>
<td>1.41 ± 0.01</td>
</tr>
<tr>
<td>P. Long</td>
<td>0.87 ± 0.02</td>
<td>0.88 ± 0.03</td>
<td>1.16 ± 0.03</td>
<td>1.190 ± 0.003</td>
</tr>
</tbody>
</table>

Note: The initial angular displacement applied to the short pendulum was 10°; for the long pendulum the initial angular displacement was 20°. The short physical pendulum needed a larger angular displacement to visually complete ten oscillations. Mathematical analysis for the long physical pendulum is outlined in Appendix H.

The data for each physical pendulum are plotted in Figs. 15 and 16. The black lines represent the theoretical maximum amplitude as a function of scaled drive frequency, using Eq. (16). The damping coefficient was experimentally determined. Each pendulum was set into motion with a stationary pivot while the motion was recorded. Using the free software Tracker [30], I measured the motion of the center of mass of each pendulum and I then fit the data to an exponentially decaying sinusoid to obtain the linear damping coefficients.
Table 12: Long physical pendulum experimental data.

<table>
<thead>
<tr>
<th>Time for 10 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.66</td>
<td>0.682 ± 0.009</td>
<td>1</td>
</tr>
<tr>
<td>13.88</td>
<td>0.720 ± 0.010</td>
<td>3</td>
</tr>
<tr>
<td>12.06</td>
<td>0.829 ± 0.014</td>
<td>4</td>
</tr>
<tr>
<td>11.21</td>
<td>0.892 ± 0.016</td>
<td>5.5</td>
</tr>
<tr>
<td>10.36</td>
<td>0.965 ± 0.019</td>
<td>7</td>
</tr>
<tr>
<td>9.69</td>
<td>1.032 ± 0.021</td>
<td>11</td>
</tr>
<tr>
<td>8.62</td>
<td>1.160 ± 0.027</td>
<td>75</td>
</tr>
<tr>
<td>7.75</td>
<td>1.290 ± 0.033</td>
<td>27</td>
</tr>
<tr>
<td>6.54</td>
<td>1.529 ± 0.04</td>
<td>6</td>
</tr>
<tr>
<td>5.51</td>
<td>1.815 ± 0.066</td>
<td>5</td>
</tr>
<tr>
<td>4.96</td>
<td>2.016 ± 0.081</td>
<td>3</td>
</tr>
<tr>
<td>9.49</td>
<td>1.053 ± 0.022</td>
<td>12</td>
</tr>
<tr>
<td>8.71</td>
<td>1.148 ± 0.026</td>
<td>86</td>
</tr>
<tr>
<td>8.46</td>
<td>1.13 ± 0.028</td>
<td>65</td>
</tr>
<tr>
<td>8.18</td>
<td>1.223 ± 0.030</td>
<td>57</td>
</tr>
<tr>
<td>7.88</td>
<td>1.269 ± 0.032</td>
<td>38</td>
</tr>
<tr>
<td>8.32</td>
<td>1.202 ± 0.029</td>
<td>72</td>
</tr>
<tr>
<td>8.21</td>
<td>1.218 ± 0.030</td>
<td>54</td>
</tr>
<tr>
<td>8.72</td>
<td>1.147 ± 0.026</td>
<td>90</td>
</tr>
</tbody>
</table>

Note: Ten complete cart oscillations were timed for each frequency.

The resonant frequency of the short physical pendulum was observed to be 1.383 ± 0.038 Hz; the long physical pendulum was observed to resonate at 1.160 ± 0.027 Hz. The resonant frequencies and the uncertainties are listed in Table 14.

The experiment clearly allows introductory students to observe a frequency dependence for oscillation amplitude. The simulated resonance curves raise questions. I generated the simulated resonance curves using the moment of inertia that can be determined using the measured small angle period of the pendulum. Even though the measured resonance curves appear to take the same shape and height as the predicted curves, there is a significant offset between the data and the theoretical curves. The offset forces the question of whether the value of $I_{CM}$ is correct.
Table 13: Short physical pendulum experimental data.

<table>
<thead>
<tr>
<th>Time for 10 cycles (s)</th>
<th>Cart Frequency (Hz)</th>
<th>Angular Displacement (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.18</td>
<td>0.550 ± 0.006</td>
<td>0.5</td>
</tr>
<tr>
<td>13.75</td>
<td>0.727 ± 0.011</td>
<td>2</td>
</tr>
<tr>
<td>11.00</td>
<td>0.909 ± 0.017</td>
<td>4.5</td>
</tr>
<tr>
<td>9.93</td>
<td>1.007 ± 0.020</td>
<td>6</td>
</tr>
<tr>
<td>9.10</td>
<td>1.099 ± 0.024</td>
<td>7</td>
</tr>
<tr>
<td>7.87</td>
<td>1.271 ± 0.032</td>
<td>20</td>
</tr>
<tr>
<td>7.23</td>
<td>1.383 ± 0.038</td>
<td>75</td>
</tr>
<tr>
<td>6.55</td>
<td>1.527 ± 0.047</td>
<td>29</td>
</tr>
<tr>
<td>6.20</td>
<td>1.613 ± 0.052</td>
<td>20</td>
</tr>
<tr>
<td>5.34</td>
<td>1.873 ± 0.070</td>
<td>10</td>
</tr>
<tr>
<td>4.76</td>
<td>2.101 ± 0.081</td>
<td>7</td>
</tr>
<tr>
<td>8.15</td>
<td>1.227 ± 0.022</td>
<td>11</td>
</tr>
<tr>
<td>7.82</td>
<td>1.279 ± 0.023</td>
<td>36</td>
</tr>
<tr>
<td>7.53</td>
<td>1.328 ± 0.025</td>
<td>88</td>
</tr>
<tr>
<td>7.24</td>
<td>1.381 ± 0.027</td>
<td>70</td>
</tr>
<tr>
<td>6.80</td>
<td>1.471 ± 0.031</td>
<td>40</td>
</tr>
<tr>
<td>7.03</td>
<td>1.423 ± 0.029</td>
<td>54</td>
</tr>
<tr>
<td>6.75</td>
<td>1.481 ± 0.031</td>
<td>34</td>
</tr>
<tr>
<td>6.52</td>
<td>1.534 ± 0.034</td>
<td>26</td>
</tr>
<tr>
<td>5.90</td>
<td>1.695 ± 0.041</td>
<td>12</td>
</tr>
</tbody>
</table>

Note: Five complete cart oscillations were timed for each frequency.

Table 14: Results for the physical pendula.

<table>
<thead>
<tr>
<th>Pendulum</th>
<th>$f_{\text{experimental}}$ (Hz)</th>
<th>$f_{\text{natural}}$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short</td>
<td>1.383 ± 0.038</td>
<td>1.399 ± 0.0003</td>
</tr>
<tr>
<td>Long</td>
<td>1.160 ± 0.027</td>
<td>1.156 ± 0.027</td>
</tr>
</tbody>
</table>

Note: The $f_{\text{natural}}$ values are determined by the stationary pivot time trials displayed in Table 11. The $f_{\text{experimental}}$ were the frequencies measured for the pendulum when it had a maximum amplitude.
Fig. 15: Amplitude as a function of cart frequency for a long physical pendula. The long physical pendulum data was listed in Table 12. The black curve represents the theoretical amplitude as a function of scaled drive frequency. The data qualitatively has the shape and maximum amplitude as the theoretical amplitude curve, but is clearly shifted to a higher frequency compared to the simulation.

Fig. 16: Amplitude as a function of cart frequency for a short physical pendula. The short physical pendulum data was listed in Table 13. The black curve represents the theoretical amplitude as a function of scaled drive frequency. The data qualitatively has the same shape and maximum amplitude as the theoretical amplitude curve, but is clearly shifted, quantitatively, to the right.
4.3 Calculations

This section outlines sample calculations of each type used to determine the results. The moment of inertia and frequency calculations for the physical pendulum are more advanced and time-consuming for introductory physics students. For upper level or graduate work, these calculations should be required. Sample calculations for the long physical pendulum are outlined in Appendix H.

The uncertainties in the calculated frequencies and times were determined by the percent uncertainty in the measured time or measured lengths. The measurement uncertainty used for length was 1 mm while the measurement uncertainty used for time was 0.2 s. An uncertainty approximation for the resonant frequency of the first simple pendulum is given by

\[ \delta f_o = \frac{\delta t}{t} f_o. \]  

Substituting values, the uncertainty in the measured resonant frequency of the first simple pendulum is

\[ \delta f_o = \frac{0.2 \text{ s}}{10.32 \text{ s}} (0.971 \text{ Hz}) \]

\[ \delta f_o = 0.02 \text{ Hz}. \]

After measuring the period of ten oscillations of the simple pendulum, I used Eq. 9 to determine a predicted period.

\[ T_o = 2\pi \sqrt{\frac{l}{g}} \]

\[ T_o = 2\pi \sqrt{\frac{0.265 \text{ m}}{9.81 \text{ m/s}}} \]

\[ T_o = 1.030 \pm 0.004 \text{ s}. \]

The uncertainties for the theoretical period and frequency of each simple pendulum were determined by propagating the measurement uncertainties with Eq. (25).
\[ \delta f_o = \sqrt{\left( \frac{\partial f_o}{\partial l} \delta l \right)^2 + \left( \frac{\partial f_o}{\partial g} \delta g \right)^2} \] (25)

For either long simple pendulum, the uncertainty calculation is:

\[ \delta f_o = \sqrt{\left( -\frac{1}{4\pi} \frac{\delta l}{\sqrt{gl^3}} \right)^2 + \left( \frac{1}{4\pi} \frac{\delta g}{\sqrt{gl}} \right)^2} \]

\[ \delta f_o = \sqrt{\left( -\frac{1}{4\pi} \frac{0.001 \text{ m}}{\sqrt{9.81 \text{ m/s}^2 (0.265 \text{ m})^3}} \right)^2 + \left( \frac{1}{4\pi} \frac{0.01 \text{ m/s}^2}{\sqrt{9.81 \text{ m/s}^2 (0.265 \text{ m})}} \right)^2} \]

\[ \delta f_o = 0.002 \text{ Hz.} \]

Propagating the uncertainties for the theoretical frequency yields a partial derivative with 12 terms because Eq. (11) will have 12 measurement uncertainties to account for. For the short physical pendulum, \( l = 5.8 \text{ cm} \) and the predicted resonant frequency will have a percent uncertainty of 1\%. The detailed calculation is in Appendix H.

Determining the moment of inertia about the center of mass of the short physical pendulum is straightforward. The total moment of inertia must be calculated and substituted into Eq. (10). Each component of the physical pendulum was analyzed. It follows that

\[ \sum I_{CM} = I_{Hub} + I_{b/s,1} + I_{b/s,2} + I_{Al,beam}. \] (26)

Using Eqs. (17–20), (10) and the values listed in Table 10, the total moment of inertia is given by

\[ \sum I_{CM} = m_{hub} \left[ \frac{r_{outer}^2}{2} + \frac{r_{inner}^2}{2} + l_{CM}^2 \right] + 2m_{b/s} \left[ l_{CM}^2 + r_{b/s}^2 \right] + \frac{m_{Al}}{12} \left[ l + l' \right]^2 + m_{Al} \left[ \frac{l - l'}{2} - l_{CM} \right]^2 \] (27)

\[ \sum I_{CM} = (4.1 \times 10^{-5} \text{ kg m}^2)_{hub} + (1.1 \times 10^{-5} \text{ kg m}^2)_{b/s} + (1.2 \times 10^{-4} \text{ kg m}^2)_{Al,beam} \]

\[ \sum I_{CM} = 1.8 \times 10^{-4} \text{ kg m}^2. \]
5 Conclusions

I have described the design, construction and testing of a new driven pendulum experiment. The purpose of the experiment is to qualitatively and quantitatively analyze a simple or physical pendulum with a horizontally driven pivot. All of the pendula that were tested had resonant frequencies $\sim 1 \text{ Hz}$ determined with an accuracy better than $0.05 \text{ Hz}$. The small oscillation natural frequencies were experimentally determined. Throughout all of the experiments, the pendula had amplitudes of oscillation that were frequency dependent and resonance was observed.

All of the design and construction goals were successfully carried out to complete the experiment. Important and successful aspects of the design were cost, ease of construction, and usability. In general, the design goal was to create a lab that can be added to any undergraduate lab course for under $200. This goal was met as long as the lab course already owns PASCO dynamics carts and tracks. The total cost of the experiment is about $147 for all of the electronics and hardware. I had the pendulum arm and square bracket professionally printed while I was constructing the experiment, which cost $49. The total cost per apparatus is $196, which does not include any deals for printing or buying parts in bulk. I can completely reconstruct the experiment using figures and descriptions from this document.

The apparatus withstood hours of use throughout experimentation. The data collection and general testing of the apparatus went smoothly with all of the pieces and components consistently working. During long sets of trials, the stepper motor would get very warm. The temperature change did not affect motor performance, but adding a heat sink onto the motor may be prudent.

All of the electrical components also consistently worked with no anomalies occurring during use. While adjusting the potentiometer that controls the frequency output of the MCU, I noticed that it is extremely sensitive; an experimenter will have to practice changing the frequencies by only fractions of a hertz. Even with its sensitivity, the MCU fulfills its two requirements; the MCU successfully allows for minor frequency changes and gives the experi-
menter the ability to start and stop the motor at a specific frequency. With the electrical set up, it is necessary to use a voltmeter or an oscilloscope to digitally monitor the output frequency of the MCU (as I discuss in Appendix C) or manually time the oscillatory frequency of the dynamics cart. I measured all of the frequencies in this experiment by timing a chosen number of oscillations of the dynamics cart.

If this experiment is being implemented into upper level labs or graduate work, an Arduino or Raspberry Pi could be substituted for the MCU. This will allow students more control over the motor and cart frequencies. It also allows for useful monitoring add-ons, like a frequency output display screen or the ability to program in specific motor frequencies. Replacing the MCU for an Arduino or Raspberry Pi should cost nearly the same, depending on what additional components are needed. An Arduino or Raspberry Pi will also work with the micro-step driver that I selected for the apparatus. In advanced physics labs, students could determine the beat frequency of a pendulum, and test what factors the beat frequencies depend on. Students can use the programs in the appendices of this document to simulate the resonance curve, or they could derive the equations of motion and program to numerically solve these equations. In advanced labs, the purpose of the experiment can be shifted to study the beats of the anharmonic oscillations of the pendulum. Students could attempt to answer questions that arose from this report, such as, why was approximating the moment of inertia for the pendulum undergoing small angle oscillations seemingly so different than the experimental data suggests? Due to the nonlinearity of the pendulum, students could attempt to build a feedback loop between the pendulum and the MCU. The pendulum could be monitored by a rotary sensor and an Arduino or Raspberry Pi could be used to modify the input signal for larger amplitudes.

The purpose of the experiment, to qualitatively and quantitatively explore resonance was at least partially achieved. Throughout the experiment, driven resonance was qualitatively observed. Some important observations that we made were resonant frequencies depend on the length of the pendula and the amplitude depends on the mass. These observations are illustrated in the Figs. 11–16. The simulations reproduce the general shape and maximum am-
plitude of the measured resonance curves. For the simple pendulum, most of the data are in quantitative agreement with the simulated resonance curves. The majority of the points near the maximum amplitude fall onto the resonance curve within uncertainty. There is a clear divergence, quantitatively, when I plot the physical pendula data with the theoretical resonance curves. The shape and maximum amplitude qualitatively match the simulated resonance curves, but the experimental frequencies are shifted to higher values than those predicted. Further experiments with controlled changes in rotational inertia should help resolve the observed discrepancy.

I found that visually measuring the amplitude of a moving pendulum is the easiest and most time efficient way to make the measurements. Using a UV trail or photogate will add time between each trial and most undergraduate students will be unable to complete the lab in a two or three hour lab period. The most important aspect of making visual measurements is to try and cut down on parallax as much as possible. Experimenters will have to align themselves with one of the maximum displacements of the pendulum and use only one eye to make measurements. This should be done when using a camera to record the amplitudes because a camera can also experience parallax. When using a camera, the uncertainties in the angle measurement will be smaller, especially if the experimenter has control over the exposure times and can reduce motion blur.

This experiment is affordable, especially when compared to most laboratory set-ups on the market. My design is a working guideline, but there are many aspects that can be adjusted according to the needs of the lab. Some of the possible design adjustments were discussed throughout this paper, but there are more possibilities. The long physical pendulum could have been converted into a dual compound pendulum with multiple pivot points; many different lengths and masses for both types of pendula could be tested, not just two lengths for each type; many variations of electronics and electrical control devices could be used, drastically changing the usability or accuracy of the experiment.

Resonance is an intriguing and powerful aspects of the physics of oscillations. It is not
common in undergraduate physics for students to get much hands-on experience with resonance experiments. This experiment provides an opportunity for students to gain such experience and illustrates how small additions of energy to an oscillating pendulum can drastically increase the oscillation amplitude of the pendulum. That is, students get to experience the magic of small oscillations forcing a system to respond with ever increasing oscillation amplitude. There exist no pendulum additions to dynamics carts in mass production. I have successfully designed, constructed and tested a new pendulum arm assembly for PASCO dynamics carts to demonstrate resonance.
References


Appendices
Appendix A  Small Angle Approximation

The small angle approximation is used by textbook authors to derive the frequency and period relationships for each type of pendulum. This is important to note because these equations are used to predict the resonant frequency for a specific pendulum. Small angular displacements must be used during the experiment if Eqs. (1), (2), (10), or (11) will be used for calculations. But what is the cutoff for a “small angle?” Recall the McLaurin series expansion for \( \sin \theta \), where \( \theta \) is measured in radians.

\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots
\]  

(28)

If \( \theta \ll 1 \text{ rad} \), then Eq. (28) shows that the second and especially the third term in the series expansion are very small when compared to the first term. With the second term falling off like the cube of the first term, it is common to show that angle measurements less than 0.2 rad (12°) will be within 1% of \( \sin \theta \). Figure 17 displays the percent difference as a function of angular displacement. The plot shows that any angle under 44° is within 10% of \( \sin \theta \). Expanding the acceptable values for the small angle approximation from 12° to 40° for this experiment does not allow us to treat the pendula as harmonic oscillators. When the drive frequency was near the resonant frequency of a pendulum, the amplitude of oscillation of the pendulum were very large. The largest angular displacement measured in this experiment was 95°.
Fig. 17: Plot of % difference versus amplitude angle for the small angle approximation. In other words, \((\theta - \sin \theta)/ \sin \theta \times 100\%\) versus \(\theta\).
Appendix B  Lagrangian Derivations

B.1 Derivation of the Equation of Motion for a Simple Pendulum Riding an Oscillating Dynamics Cart

The simple pendulum is illustrated in Fig. 1. The horizontal position of the oscillating pivot is given by \( X(t) = A_p \cos \omega_D t \) and the position \((x, y)\) of the center of mass of the pendulum bob is specified by Eqs. (3) and (4). The position functions are differentiated with respect to time and substituted into the energy functions, \( T(\theta, \dot{\theta}, t) \) and \( V(\theta) \).

\[
x \left( \theta, \dot{\theta}, t \right) = l \dot{\theta} \cos \theta - A_p \omega_D \sin \omega_D t
\]

\[
\dot{y} \left( \theta, \dot{\theta}, t \right) = l \dot{\theta} \sin \theta
\]

\[
T \left( \theta, \dot{\theta}, t \right) = \frac{m}{2} \left[ \dot{x}_{CM}^2 + \dot{y}_{CM}^2 \right]
\]

\[
T \left( \theta, \dot{\theta}, t \right) = \frac{m}{2} \left[ l^2 \dot{\theta}^2 - 2A_p \omega_D l \dot{\theta} \sin \omega_D t \cos \theta + A_p^2 \omega_D^2 \sin^2 \omega_D t \right]
\]

Recall that the potential energy of the pendulum bob is

\[
V(\theta) = mgl \left( 1 - \cos \theta \right).
\]

Substituting Eqs. (32) and (33) into Eq. (5) yields the Lagrangian for the simple pendulum with an oscillating pivot.

\[
L(\theta, \dot{\theta}, t) = \frac{ml^2}{2} \left[ \dot{\theta}^2 - \left( \frac{A_p \omega_D}{l} \right) 2 \dot{\theta} \cos \theta \sin \omega_D t + \left( \frac{A_p \omega_D}{l} \right)^2 \sin^2 \omega_D t - \left( \frac{2g}{l} \right) \left( 1 - \cos \theta \right) \right]
\]

Now the partial derivatives of the Lagrangian with respect to \( \theta \) and \( \dot{\theta} \) are derived.

\[
\frac{\partial L}{\partial \theta} = \dot{\theta} ml^2 - A_p \omega_D ml \sin \omega_D t \cos \theta
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \ddot{\theta} ml^2 + \dot{\theta} A_p \omega_D ml \sin \omega_D t \sin \theta - A_p \omega_D^2 ml \cos \omega_D t \cos \theta
\]  
\quad (35)

\[
\frac{\partial L}{\partial \theta} = A_p \omega_D ml \dot{\theta} \sin \omega_D t \sin \theta - mgl \sin \theta
\]  
\quad (36)

Equations (35) and (36) are substituted into Eq. (7) and the undamped equation of motion for the simple pendulum is derived:

\[
\ddot{\theta} ml^2 + mgl \sin \theta - A_p \omega_D^2 ml \cos \omega_D t \cos \theta = 0.
\]  
\quad (37)

Dividing by \( ml^2 \) yields

\[
\ddot{\theta} - \frac{A_p}{l} \omega_D^2 \cos \theta \cos \omega_D t + \frac{g}{l} \sin \theta = 0.
\]  
\quad (8)

**B.2 Derivation of the Equation of Motion for a Physical Pendulum Riding an Oscillating Dynamics Cart**

The physical pendulum is illustrated in Fig. 2. The pivot point has the same motion as described for the simple pendulum. The position functions of the center of mass (CM) of the physical pendulum are given by Eqs. (12) and (13). The position functions are differentiated with respect to time and the energy functions for the CM of the pendulum are determined. The kinetic energy function is displayed in Eq. (38), and is the sum of the translational and rotational kinetic energies.

\[
T(\theta, \dot{\theta}, t) = \frac{\sum I \dot{\theta}^2 + M}{2} \left[ \dot{x}^2 + \dot{y}^2 \right]
\]  
\quad (38)

\[
T(\theta, \dot{\theta}, t) = \frac{\sum I \dot{\theta}^2 + M}{2} \left[ \dot{x}^2 \dot{\theta} + 2A_p \omega_D l_{CM} \dot{\theta} \sin \omega_D t \sin \theta + A_p^2 \omega_D^2 \sin \omega_D t \right]
\]  
\quad (39)

The potential energy function, \( V(\theta) \), is the same as defined for the simple pendulum, displayed in Eq. (33). Substituting the energy functions into Eq. (5), the Lagrangian for the undamped physical pendulum with an oscillating pivot point is determined.
\[ L(\theta, \dot{\theta}, t) = \left[ \sum I + Ml_{CM}^2 \right] \dot{\theta}^2 + \left[ \frac{MAp\omega_D \sin \omega_D t}{2} \right] \left( A_p\omega_D \sin \omega_D t - 2l_{CM} \dot{\theta} \cos \theta \right) - Mgl_{CM} (1 - \cos \theta) \]  

(14)

\[ \frac{\partial L}{\partial \dot{\theta}} = \left[ I + Ml_{CM}^2 \right] \dot{\theta} + \left[ \frac{MAp\omega_D \sin \omega_D t}{2} \right] (-2l_{CM} \cos \theta) \]  

(40)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \left[ I + Ml_{CM}^2 \right] \ddot{\theta} - [MAP\omega_{Dl_{CM}}] \left( \omega_D \cos \omega_D \cos \dot{\theta} \sin \omega_D t \sin \theta \right) \]  

(41)

\[ \frac{\partial L}{\partial \theta} = MAP\omega_{Dl_{CM}} \dot{\theta} \sin \omega_D t \sin \theta - Mgl_{CM} \sin \theta \]  

(42)

Substituting Eqs. (41) and (42) into (7) yields the equation of motion for the undamped physical pendulum.

\[ \left[ I + Ml_{CM}^2 \right] \ddot{\theta} - MAP\omega_{Dl_{CM}} \cos \theta \cos \omega_D t + Mgl_{CM} \sin \theta = 0 \]  

(15)
Appendix C  Cart Frequency versus MCU Output Voltage

We expect that the rotational speed of the motor and, therefore, the frequency of the oscillations of the pivot to depend linearly on the output signal of the MCU. The output signal of the MCU can be monitored with a voltmeter or an oscilloscope. Even though the output signal is DC, it is a PWM signal (see Section 2.1) so it can be monitored with a voltmeter or with an oscilloscope, on which the voltage will appear as a triangle wave.

I connected an oscilloscope to the PUL+ terminal on the micro-step driver and grounded the oscilloscope to the common anode on the MCU, to monitor the output frequency of the MCU. This connection allowed me to measure the output signal with a small uncertainty in the frequency that I could convert directly into motor speed and cart frequency. The frequency of the output signal should be thought of as “steps per second.” For example, if the output frequency is 1000 Hz, this is the same as 1000 steps per second. If we multiply the measured frequency of the output signal by the \(0.9 \text{ deg/step}\) of the motor, we get 900 \(\text{deg/s}\). If we convert the degrees to revolutions, we get \(\left[900 \text{ deg/s}\right] \left[\frac{1 \text{ rev}}{360 \text{ deg}}\right] = 2.5 \text{ rev/s}\). So if the measured frequency on the oscilloscope is 1000 Hz, the cart will have a period of 0.4 s.

I measured the oscillation period of the cart as a function of MCU output voltage. The measured values are shown in Table 15. I determined the oscillation frequency from the period using the relation \(f_D = \frac{1}{T_D}\). The plot of \(f_D\) versus MCU output voltage is shown in Fig. 18. The voltage changes in the MCU output voltage are in the millivolt range, so to get an accurate relationship, a voltmeter with millivolt resolution must be used. I used a voltmeter with an average uncertainty of 10 mV so the relationship obtained from the data has some uncertainty. A linear, least-squares fit yields a slope of \(3.563 \frac{\text{Hz}}{V}\) and \(R^2\) value of 0.999.
Table 15: Cart frequency versus MCU output voltage data.

<table>
<thead>
<tr>
<th>Voltage (V)</th>
<th>$T_D$ (s)</th>
<th>$f_D = \frac{1}{T_D}$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.37±0.01</td>
<td>3.46±0.04</td>
<td>0.289±0.003</td>
</tr>
<tr>
<td>8.38±0.01</td>
<td>3.12±0.04</td>
<td>0.320±0.004</td>
</tr>
<tr>
<td>8.41±0.01</td>
<td>2.34±0.04</td>
<td>0.428±0.007</td>
</tr>
<tr>
<td>8.43±0.01</td>
<td>1.99±0.04</td>
<td>0.502±0.01</td>
</tr>
<tr>
<td>8.44±0.01</td>
<td>1.86±0.04</td>
<td>0.538±0.012</td>
</tr>
<tr>
<td>8.47±0.01</td>
<td>1.62±0.04</td>
<td>0.618±0.015</td>
</tr>
<tr>
<td>8.49±0.01</td>
<td>1.44±0.04</td>
<td>0.696±0.019</td>
</tr>
<tr>
<td>8.50±0.01</td>
<td>1.36±0.04</td>
<td>0.735±0.022</td>
</tr>
<tr>
<td>8.51±0.01</td>
<td>1.28±0.04</td>
<td>0.781±0.024</td>
</tr>
<tr>
<td>8.53±0.01</td>
<td>1.22±0.04</td>
<td>0.821±0.027</td>
</tr>
<tr>
<td>8.54±0.01</td>
<td>1.14±0.04</td>
<td>0.880±0.031</td>
</tr>
<tr>
<td>8.56±0.01</td>
<td>1.10±0.04</td>
<td>0.910±0.033</td>
</tr>
<tr>
<td>8.58±0.01</td>
<td>0.990±0.04</td>
<td>1.007±0.041</td>
</tr>
<tr>
<td>8.60±0.01</td>
<td>0.900±0.04</td>
<td>1.111±0.049</td>
</tr>
<tr>
<td>8.63±0.01</td>
<td>0.850±0.04</td>
<td>1.182±0.056</td>
</tr>
<tr>
<td>8.65±0.01</td>
<td>0.800±0.04</td>
<td>1.247±0.062</td>
</tr>
<tr>
<td>8.69±0.01</td>
<td>0.700±0.04</td>
<td>1.430±0.082</td>
</tr>
<tr>
<td>8.715±0.01</td>
<td>0.660±0.04</td>
<td>1.512±0.092</td>
</tr>
<tr>
<td>8.735±0.01</td>
<td>0.630±0.04</td>
<td>1.592±0.101</td>
</tr>
<tr>
<td>8.76±0.01</td>
<td>0.600±0.04</td>
<td>1.675±0.112</td>
</tr>
<tr>
<td>8.815±0.01</td>
<td>0.540±0.04</td>
<td>1.859±0.138</td>
</tr>
<tr>
<td>8.875±0.01</td>
<td>0.480±0.04</td>
<td>2.083±0.174</td>
</tr>
</tbody>
</table>

Fig. 18: Plot of the cart frequency versus the output voltage of the MCU. The cart frequency depends linearly on the output voltage of the MCU, as expected.
I show the code that I developed for the printed components of the apparatus in this Appendix. All of the numbers in the code are in units of millimeters.

D.1 Pendulum Arm Code

\textit{Written by:}

Ryan M. Pacheco

Department of Physics and Astronomy

Eastern Michigan University

Ypsilanti, MI 48197

rpachec1@emich.edu

\begin{verbatim}
union(){
  polyhedron( points = [ [-35, 0, 6], [35, 0, 6], [15,310,6], [-15,310,6], [-15,310,-0], [15,310,-0], [35,0,-0], [-35,0,-0], [-25, 15, 6], [25, 15, 6], [5,295,6], [-5,295,6], [-5,295,-0], [5,295,-0], [25,15,-0], [-25,15,-0] ], faces = [ [0,3,11,8], [0,8,9,1], [1,9,10,2], [2,10,11,3], [3,0,7,4], [0,1,6,7], [1,2,5,6], [2,3,4,5], [10,13,12,11], [13,10,9,14], [14,9,8,15], [15,8,11,12], [12,13,5,4], [13,14,6,5], [14,15,7,6], [15,12,4,7] ], convexity = 10
); // This is the base shape, a hollow trapazoidal polyhedron
\end{verbatim}

53
$fn=100;
translate([0,310,0]){
cylinder(h=20, r=15.7, center=false);
}
// Larger top cylinder

translate([0,310,25]){
difference(){
cylinder(h=35, r=3.175, center=true);
translate([0,0,16]){cube([1,16,10],true);}}}
}
// Long top cylinder, with slit cut out

translate([0,2,16]){
cube([70,4,20],true);
}
// Bottom bracket

// below are adding gussets
polyhedron( points = [ [31.5,4,13], [31.5,4,6], [18,197,6], [15,197,6], [28.5,4,6], [28.5,4,13] ],
faces = [ [0,1,2], [2,3,5,0], [2,3,4,1], [0,1,4,5], [5,3,4] ], convexity = 10);
polyhedron( points = [ [-28.5,4,13], [-28.5,4,6],...
D.2 Square Bracket Code

// Written by:

//

// Ryan M. Pacheco
// Department of Physics and Astronomy
// Eastern Michigan University
// Ypsilanti, MI 48197
// rpacheco1@emich.edu

// All numbers are in mm
difference(){
cube([52.8,133.5,8 ],true);
  // Base cube
  $fn=100;
  // Iterations for cylinders
  translate([15.5,45.5,0]){
    cylinder(h=10,r=3.1, center=true);
  }
  // Screw hole
  translate([-15.5,-45.5,0]){
    cylinder(h=10,r=3.1, center=true);
  }
  // Screw hole
  translate([-25,0,-2]){
    cube([50,70,4.1],true);
  }
}
// Bottom support Slot
translate([-27.5,0,0]){
cube([15,70,10],true); }

// Verticle opening for pendulum bracket
translate([10,-20,0]){
cube([33.1,100,10],true); }

// Cut out to make original cube an L shape
polyhedron( points = [ [-13.4,28,0], [-13.4,27.147,4.05],
[-13.4,31.147,4.05], [-13.4,32,0], [-20.4,32,0],
[-20.4,28,0], [-20.4,27.147,4.05], [-20.4,31.147,4.05]],
faces = [ [0,1,2,3], [3,4,5,0], [0,5,6,1], [1,6,7,2],
[2,7,4,3], [4,7,6,5] ], convexity = 10 );

// Slit for gusset
polyhedron( points = [ [-13.4,-28,0], [-13.4,-27.147,4.05],
[-13.4,-31.147,4.05], [-13.4,-32,0], [-20.4,-32,0],
[-20.4,-28,0], [-20.4,-27.147,4.05], [-20.4,-31.147,4.05]],
faces = [ [0,1,2,3], [3,4,5,0], [0,5,6,1], [1,6,7,2],
[2,7,4,3], [4,7,6,5] ], convexity = 10 );

// Slit for gusset
Fig. 19: A photograph of the entire apparatus.
Fig. 20: A photograph of the 3D-printed components. The square bracket is bolted to the dynamics cart and the pendulum arm is attached and standing vertically.
Fig. 21: A photograph of a simple pendulum attached to the apparatus. The 26.5 cm, 20 g simple pendulum is attached to the pendulum arm.
Fig. 22: Photographs of the physical pendula. A photograph of the short physical pendulum on the left and of the long physical pendulum on the right.
Fig. 23: A photograph of the MSD and MCU. In this photograph the MSD and MCU are attached with a *common anode* connection.
Appendix F Approximating a Damping Constant

The damping constant that I used for each numerical integration was determined with the same method. A pendulum was allowed to oscillate with an initial angular displacement of $5^\circ$ and a stationary pivot. The motion was recorded until the pendulum stopped oscillating and then I tracked the motion using Tracker software [30]. The Tracker data was plotted and I used a Locally Weighted Scatter Plot Smoothing (LOWESS) on qtiplot [31] to smooth and center the data. I plot the angular displacement as a function of time in Fig. 24.

![Angular position of a damped simple pendulum](image)

**Fig. 24:** Plot of the position of a damped simple pendulum with a stationary pivot.

The equation of motion for the simple pendulum will have an exponential and sinusoidal solution such that

$$
\theta(t) = Ae^{B(t-C)} \sin D(t - E) + F.
$$

I used this general form to plot another set of data on top of the angular position of the pendulum. I then minimized the difference of the sum of the squares of the two sets of data. I display the fit data plotted over the angular position data in Fig. 25. For the displayed data, the difference between the sum of the squares was 0.00057 and was comparing 5284 different datum.

By minimizing the difference of the sum of the squares I determined $B = -0.005707615 \text{ rad/s}$
and for the simple pendulum, \( B = \frac{b}{2m} \), which is the damping constant [32]. I used this approximation of the damping constant during the numerical integration of the equation of motion for the 26.5 cm simple pendulum with a moving pivot. I used this approximation aware that it could be inaccurate for the pendulum with a moving pivot. Future studies may attempt to use a non-linear damping function.

Fig. 25: Plot of the position of a damped pendulum with fit data. Plot with the fit data superposed on top.
Appendix G  Python Code

G.1  Maximum Amplitude as a function of Drive Frequency: Numerical Integration and Plotting
Code

This is the code we used to numerically integrate the equation of motion of each pendulum. The specific code given is for the long, 20 g simple pendulum. The plot that this code yielded is shown in Fig. 11.

# Simple_Pendulum_w_Osc_Pivot_Res_Curve_v1.py
#
# A simple pendulum is a point mass
# attached by an ideal string to a pivot.
# Here, the pivot is sinusoidally driven.
#
# We assume that there is no drag or friction.
#(We now have a damping constant – RP)
#
# This file will generate a plot of
# the maximum amplitude of oscillation versus drive frequency,
# i.e., the resonance curve.
#   (File now plots data over the numerical integration – RP)
# The general idea: start the pendulum from rest and let it go
# through 30 natural time periods worth of driven oscillation.
#For example, if the drive period is twice as long as the natural
# period then the time window is 15 drive cycles in duration.
#
# Written by:
#
# Ernest R. Behringer
# import the commands needed to make the plot
from pylab import plot,xlabel,ylabel,grid,show,figure,xlim,ylim,title

# import the command needed to make a 1D array
from numpy import array,arange,pi,sqrt,sin,cos,zeros,exp,vstack,linspace

from scipy.integrate import odeint
from matplotlib.pyplot import errorbar
# Inputs

g = 9.81 # gravitational acceleration [m/s^2]
L = 0.165 # distance from pivot to CM of simple pendulum bob [m]
mass = 0.020 # mass of the pendulum bob [kg]
X_0 = 0.013 # amplitude of the pivot motion [m]
theta_i = 0.0 # initial angular position WRT the vertical [rad]
omega_i = 0.0 # initial angular speed [rad/s]
counter = 0 # counter for the loop
beta_0 = 0.11 # constant in front of dx [Hz]

# Define Data:

#Data from 20 g, 16.5 cm simple pendulum (first set)
data_short_20g_amplitude = [0.052359878, 0.06981317, 0.078539816, 0.06981317, 0.078539816, 0.104719755, 0.20943951, 1.082104136, 0.20943951, 0.165806279, 0.113446401, 0.087266463, 0.06981317, 0.06981317, 0.052359878]
data_short_20g_amplitude_unc = [0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03]
data_short_20g_scaledFreq = [0.547, 0.598, 0.676, 0.746, 0.790, 0.852, 0.874, 0.984, 1.204, 1.264, 1.379, 1.529, 1.730, 1.843, 1.878]
data_short_20g_scaledFreq_unc = [0.00645586, 0.007049678, 0.007976153, 0.009946732, 0.011612267, 0.013271163, 0.014689804, 0.016971716, 0.023381461, 0.03002856, 0.034400064, 0.041612144, 0.052192706, 0.062915935, 0.068275441]

#Data from 20 g, 16.5 cm simple pendulum (second set)
data_short_20g_amplitude2 = [0.750492, 0.401426, 0.331613, 0.523599, 0.959931, 0.872665, 0.436332, 0.715585, 0.890118, 1.029744, 1.082104, 1.483530, 1.658063, 0.645772, 0.209440]
data_short_20g_amplitude_unc2 = [0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03, 0.03]
data_short_20g_scaledFreq2 = [1.044866571, 1.081757267, 1.151243188, 1.050252481, 1.007411527, 1.001223495, 1.063963349, 1.029035259, 1.006167809, 0.99389747, 0.97956241, 0.94766968, 0.905551028, 0.885865136, 0.867016942]
data_short_20g_scaledFreq_unc2 = [0.053582901, 0.057433356, 0.065035265, 0.054136726, 0.049810211, 0.049200172, 0.055559444, 0.051971478, 0.049687299, 0.048482803, 0.047094347, 0.04407766, 0.040246712, 0.038515875, 0.036894338]

# Calculate quantities [SI units]:
omega_sa = sqrt(g/L) # angular frequency of small amplitude oscillations [rad/s]
period_sa = 2.0*pi/omega_sa # small amplitude period [s]
omega_D_lo = 0.5*omega_sa # low angular drive frequency [rad/s]
omega_D_hi = 1.5*omega_sa # high angular drive frequency [rad/s]
delta_omega_D = 0.01*omega_sa # interval of angular drive frequency [rad/s]

# array of drive frequencies
omega_D = arange(omega_D_lo,omega_D_hi,delta_omega_D)
amplitudes = zeros(len(omega_D))

print("Small amplitude period is %s s." %period_sa)
print("The initial angular position is %s rad." %theta_i)
print("The initial angular velocity is %s rad/s." %omega_i)

# Define omega_D_temp for now
omega_D_temp = omega_D[0]

# Here are the derivatives
def derivs(r,t):
    x = r[0]
    xp = r[1]
    dx = xp
    ddx = omega_D_temp*omega_D_temp*X_0*cos(omega_D_temp*t)*cos(x)/L - (g/L)*sin(x)-beta_0*dx
    return array([dx,ddx],float)

# Specify the initial and final time for the integration
    t1 = 0.0 # initial scaled time
    t2 = 40.0*period_sa # final scaled time
    N = 8000 # number of time steps
    h = (t2-t1)/N # time step size
    tpoints = arange(t1,t2,h)
    print len(tpoints)

# Start the loop over angular drive frequencies
for n in omega_D:
    omega_D_temp = n
# Specify initial conditions
x_i = theta_i # theta = theta0 is at the beginning of the motion
dxdt_i = omega_i # dxdt0 is the velocity at the start gate [m/s]
r_i = array([x_i,dxdt_i],float)
# Integrate using the fourth-order Runge-Kutta method with
# fixed stepsize
r = odeint(derivs, r_i, tpoints)
# Save the maximum amplitude of the solution
amplitudes[counter] = max(r[:,0])
# Increment the counter
counter = counter + 1
# Plot the resonance curve
figure("Resonance Curve for the Simple Pendulum with Oscillating Pivot")
plot(omega_D/omega_sa, amplitudes, "k-")
xlim(omega_D_lo/omega_sa, omega_D_hi/omega_sa)
ylim(0.0, max(amplitudes))
xlabel("Scaled Angular Drive Frequency $\omega_D/\omega_o$", fontsize=16)
ylabel("Oscillation Amplitude $\theta$ [rad]", fontsize=16)
grid(True)
# Add data set 1 to plot
errorbar(data_short_20g_scaledFreq, data_short_20g_amplitude,
xerr=data_short_20g_scaledFreq_unc,
yerr=data_short_20g_amplitude_unc,
fmt="b.", label="Measured Amplitudes")
# Add data set 2 to plot
errorbar(data_short_20g_scaledFreq2, data_short_20g_amplitude2, xerr=data_short_20g_scaledFreq_unc2, yerr=data_short_20g_amplitude_unc2, fmt="b.", label="Measured Amplitudes")
title("Length = %s m, $\omega_{o}$ = %.2f rad/s, $T_{o}$ = %.2f s" 
(L, omega_sa, period_sa))
show()

G.2 Amplitude Evolution with Time for a Pivot Driven Simple Pendulum, with Specific $\omega_{D}$: Python Plot Code

This is the code we used to predict the amplitude as a function of time evolution for a pendulum with a horizontally oscillating, driven pivot. The plot yielded from this code is shown in Fig. 14.

#
# Simple_Pendulum_w_Osc_Pivot_Evolution_v1.py
#
# A simple pendulum is a point mass
# attached by an ideal string to a pivot.
# Here, the pivot is sinusoidally driven.
#
# We assume that there is no drag or friction.
# A damping constant has been added (RP)
#
# This file will generate a plot of
# the angular position versus time.
#
# Written by:
# Ernest R. Behringer
# Department of Physics and Astronomy
# Eastern Michigan University
# Ypsilanti, MI 48197
# (734) 487-8799
# ebehringe@emich.edu
#
# 20180918 by ERB
#
# Edited by:
#
# Ryan Pacheco
# In attempts to complete a thesis
# rpachec1@emich.edu
# rmpachec@umich.edu
# October 27, 2018
# import the commands needed to make the plot
from pylab import plot,xlabel,ylabel,grid,show,figure,xlim,ylim,
title
# import the command needed to make a 1D array
from numpy import array,arange,pi,sqrt,sin,cos, exp
from scipy.integrate import odeint import odeint
# Inputs
g = 9.8 # gravitational acceleration [m/s²]
L = 0.265 # distance from pivot to CM of simple pendulum bob [m]
mass = 0.200 # mass of the pendulum bob [kg]
X_0 = 0.013 # amplitude of the pivot motion [m]
theta_i = 0.0 # initial angular position WRT the vertical [rad]
omega_i = 0.0 # initial angular speed [rad/s]
beta_0 = 0.3 # constant in front of e in beta [rad/s]

# Calculate quantities [SI units]:
omega_sa = sqrt(g/L) # angular frequency of small amplitude oscillations
period_sa = 2.0*pi/omega_sa # small amplitude period
omega_D = 1*omega_sa # angular drive frequency
print("Small amplitude period is %s s." %period_sa)
print("The initial angular position is %s rad." %theta_i)
print("The initial angular velocity is %s rad/s." %omega_i)

# Here are the derivatives

def derivs(r,t):
x = r[0]
xp = r[1]
dx = xp
ddx = omega_D*omega_D*X_0*cos(omega_D*t)*cos(x)/L - (g/L)*sin(x)
    #-beta_0*dx
return array([dx,ddx],float)

# Specify initial conditions
x_i = theta_i # theta = theta0 is at the beginning of the motion
dxdt_i = omega_i # dxdt0 is the velocity at the start gate [m/s]
r_i = array([x_i,dxdt_i],float)

# Calculate the numerical solution using
# fourth-order Runge-Kutta algorithm
t1 = 0.0 # initial scaled time
t2 = 30.0*period_sa # final scaled time
N = 6000 # number of time steps
h = (t2-t1)/N # time step size
tpoints = arange(t1,t2,h)
# print len(tpoints)

r = odeint(derivs,r_i,tpoints)
# print len(r[:,0])
# print max(r[:,0])

# Plot the position versus time
figure()
plot(tpoints,r[:,0],"b-")
xlim(t1,t2)
# ylim(x0,max(r[:,0]))
ylim(-0.5*pi,0.5*pi)
xlabel("Time $t$ [s]",fontsize=16)
ylabel("Angular position $\theta$ [rad]",fontsize=16)
grid(True)
title("Length = %s m, Drive Frequency $\omega_D$ = %.2f rad/s, $\omega_0$ = %.2f rad/s"
(L,omega_D,omega_sa))
show()

# Plot the angular velocity versus time
figure()
plot(tpoints,r[:,1],"g-")
```python
xlim(t1,t2)
#ylim(0,max(r[:,1]))
ylim(min(r[:,1]),max(r[:,1]))
xlabel("Time $t$ [s]",fontsize=16)
ylabel("Angular velocity $\omega$ [rad/s]",fontsize=16)
grid(True)
title("Length = %s m, Drive Frequency $\omega_D$ = %.2f rad/s, $\omega_0$ = %.2f rad/s"% (L,omega_D,omega_sa))
show()

# Plot the angular velocity versus angular position (phase space plot)
figure()
plot(r[:,0],r[:,1],"r-")
xlim(-theta_i,theta_i)
#ylim(0,max(r[:,1]))
ylim(min(r[:,1]),max(r[:,1]))
xlabel("Angular position $\theta$ [rad]",fontsize=16)
ylabel("Angular velocity $\omega$ [rad/s]",fontsize=16)
grid(True)
title("Length = %s m, Drive Frequency $\omega_D$ = %s rad/s, $\omega_0$ = %s rad/s"% (L,omega_D,omega_sa))
show()
```
H.1 Uncertainty in the Theoretical Resonant Frequency of the Short Physical Pendulum

The partial derivative of Eq. (11), with respect to twelve measured quantities, must be determined. The total partial derivative is shown symbolically in Eq. (44) and each individual partial derivative is listed.

\[
\delta f_o = \sqrt{\sum_{i=1}^{12} \left( \frac{\partial f_o}{\partial x_i} \delta x_i \right)^2} \quad \text{(44)}
\]

\[
\frac{\partial f_o}{\partial g} = -\frac{1}{4\pi} \sqrt{\frac{\sum I}{Ml_{CM}^3g^3}} \\
\frac{\partial f_o}{\partial l_{CM}} = -\frac{1}{4\pi} \sqrt{\frac{\sum I}{Ml_{CM}^3g}} \\
\frac{\partial f_o}{\partial M} = -\frac{1}{4\pi} \sqrt{\frac{\sum I}{M^3l_{CM}g}} \\
\frac{\partial f_o}{\partial m_b} = \frac{1}{4\pi} \sqrt{\frac{2r_b^2}{Mm_bgl_{CM}}} \\
\frac{\partial f_o}{\partial r_b} = \frac{1}{2\pi} \sqrt{\frac{2m_b}{Mgl_{CM}}} \\
\frac{\partial f_o}{\partial l'} = \frac{1}{2\pi} \sqrt{\frac{m_H}{2Mgl_{CM}}} \\
\frac{\partial f_o}{\partial m_H} = \frac{1}{4\pi} \sqrt{\frac{r_o^2 + r_i^2}{Mm_Hgl_{CM}}} \\
\frac{\partial f_o}{\partial r_o} = \frac{1}{2\pi} \sqrt{\frac{m_H}{Mgl_{CM}}} \\
\frac{\partial f_o}{\partial r_i} = \frac{1}{2\pi} \sqrt{\frac{m_H}{2Mgl_{CM}}} \\
\frac{\partial f_o}{\partial m_A} = \frac{1}{4\pi} \sqrt{\frac{l^3 + l^3}{3LMm_Agl_{CM}}} \\
\frac{\partial f_o}{\partial m_A} = \frac{3}{4\pi} \sqrt{\frac{m_Hl'}{3LMgl_{CM}}} \\
\frac{\partial f_o}{\partial l} = \frac{3}{4\pi} \sqrt{\frac{m_Al'}{3LMgl_{CM}}}
\]

Solving each of the partial derivatives using values contained in Table 9 and solving Eq. (44) yields the uncertainty in the predicted resonant frequency of the short physical pendulum.

\[
\delta f_o = 0.003 \text{ Hz}
\]
H.2 Moment of Inertia Calculation for the Long Physical Pendulum

To construct the long physical pendulum I attached the second aluminum beam to the first with two bolts and nuts effectively extending the length of the physical pendulum. I display a photograph of the physical pendula in Appendix E. I approximated the moment of inertia for the long physical pendulum using Eqs. (17–20), (10) and the values listed in Table 10.

\[ \sum I_{CM} = I_{hub} + I_{bolt/\text{spacer},1} + I_{bolt/\text{spacer},2} + I_{bolt/nut,1} + I_{bolt/nut,2} + I_{Al,\text{beam},1} + I_{Al,\text{beam},2} \quad (45) \]

\[ \sum I_{CM} = m_{hub} \left[ \frac{r_o^2}{2} + \frac{r_i^2}{2} + l_{CM}^2 \right] + 2m_{b/s} \left[ r_{b/s}^2 + l_{CM}^2 \right] + m_{b/n} \left[ l - r_{b,1} \right]^2 + m_{b/n} \left[ l - l_{CM} \right]^2 + \frac{m_{Al,1}}{12} L_1^2 + m_{Al,1} \left[ l_{CM} + l' - \frac{l}{2} \right]^2 + \frac{m_{Al,2}}{12} L_2^2 + m_{Al,2} \left[ \frac{L_2}{2} - l_{CM} - r_{b,1} \right]^2 \quad (46) \]

Substituting values from Table 10 yields a moment of inertia for the long physical pendulum.

\[ \sum I_{CM} = 1.4 \times 10^{-4} \text{kg m}^2 + 4.0 \times 10^{-5} \text{kg m}^2 + 3.3 \times 10^{-5} \text{kg m}^2 + 1.6 \times 10^{-5} \text{kg m}^2 + 1.2 \times 10^{-4} \text{kg m}^2 + 3.1 \times 10^{-4} \text{kg m}^2 \]

\[ \sum I_{CM} = 6.6 \times 10^{-4} \text{kg m}^2 \]