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FRACTAL PATTERNS IN CHAOTIC SCATTERING

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ABSTRACT

Fractal patterns can be observed in chaotic scattering, as occurs when light scatters from arrays of cylindrical or spherical mirrors. We provide an introduction to these optical systems and the corresponding fractal patterns generated by their chaotic behavior. We also describe the theoretical and computational models used to simulate the behavior of these systems, and the design of experimental apparatus to check the models.

INTRODUCTION

A fractal pattern is a geometric figure that is self-similar: it looks much the same whether viewed close up or from far away. Such fractal patterns can be produced in physical systems that display chaotic scattering. Chaotic scattering occurs when an incident particle interacts with a collection of particles (the scatterers) and is redirected in a way that is highly sensitive to the conditions of incidence. When predicting the trajectory of a single incident particle, straightforward rules are applied. However, when predicting the trajectories of a collection of incident particles, it is possible to observe complex behavior that emerges from the straightforward rules. Two-dimensional systems that display chaotic scattering of light include a pair of cylindrical mirrors (“two disc system”), and an array of three circular mirrors whose centers form an equilateral triangle (“three disc system”). A three-dimensional system that displays chaotic scattering is a tetrahedral stack of spherical mirrors. We also discuss these systems below.
Regardless of the system, each incident light ray produces a reflected ray that can be described by the angle it makes with the incident ray, known as the *scattering angle*. If the scattering angle of the incident ray is highly sensitive to the characteristics of the incident ray, then the scattering is chaotic. Within our systems, the key characteristic of each incident ray is its impact parameter, which is the shortest distance from the line containing the incident ray to the center of the system, as shown in Fig. 1., for a single disc system.

![Figure 1](image)

*Figure 1.* The impact parameter for a single disc system.

It is important to note that chaotic systems, in general, are also deterministic systems and are not necessarily random systems (2). This means that knowledge of the incident conditions should be sufficient to predict the outcome of a scattering event. The term “chaos” is used to describe the high sensitivity to the initial conditions.

Within the optical systems described above that display chaotic behavior, we can find or observe self-similar fractals. The two-dimensional systems described above produce fractals in graphs of scattering angle versus impact parameter. The three-dimensional tetrahedral stack of spherical mirrors produces a fractal
pattern as described by Sweet et al. (1) and is shown in Fig. 2.

The long-term goals of this project include the creation of computational models to simulate chaotic scattering and generate the corresponding fractal patterns, and the observation of fractal patterns in the laboratory through the construction of an appropriate experimental apparatus. We will describe the dependence of the observed fractal pattern on the characteristics of the incident rays. The short-term goals of this project are to create computational models to simulate chaotic scattering in the two-dimensional systems described above.

The remainder of this paper is organized as follows: (1) We describe the theoretical and computational models of scattering from cylindrical or spherical mirrors; (2) we describe the experimental apparatus that can be constructed to test these models; and (3) we conclude with a summary of our next steps.

I. Theoretical And Computational Models

We will consider four different systems in order to build our understanding of chaotic scattering. the single, two-, and three-disc systems and the tetrahedral stack of spheres.

The single disc system consists of a single reflecting disc, as shown in Fig. 1. (above), and is not expected to display any chaotic behavior. The law of reflection is used to predict the scattering angle of each incident ray. The law of reflection states that a light ray incident upon a mirror will be reflected at an angle equal to the angle of incidence that is measured, relative to the line perpendicular to the surface (called the surface normal). Applying
the law of reflection and geometry, we can construct a plot of the scattering angle versus impact parameter, as shown in Fig. 3. This relationship holds for all single disc reflections and can be used for systems with more than one disc.

![Figure 3. Scattering angle versus scaled impact parameter. The scaled impact parameter is the impact parameter divided by the radius R of the disc (b/R). Current work.](image)

The more complicated two-dimensional system is the two-disc system. When an incident ray enters the two-disc system, there are three possible outcomes: first, rays pass straight through; second, a ray may scatter out of the system; and third, the ray becomes trapped within the system. We will focus on the case in which a ray scatters from the system, calculating where the rays go and how many reflections it takes for a ray to reflect out of the system. We do not expect to observe fractal patterns like those observed in the tetrahedral stack of mirrored spheres, because the scattering is confined to a plane (so two-dimensional images are not formed). Nonetheless, computational models are expected to show chaotic scattering. The deflection function shown in Fig. 3, can be applied
to each encounter of a light ray with a reflecting disc until either the light ray leaves the two-disc array or is trapped in the array.

The final two-dimensional system we consider is the three-disc system, which also displays chaotic behavior [3]. This system consists of three identical reflective cylindrical discs arranged so that their centers form an equilateral triangle. Gaps exist between the edges of the discs so that light may enter the region between the discs. Depending on the ratio of the gap size to the diameter of the discs, the path of a light ray can change dramatically.

![Figure 4](image.png)

**Figure 4.** Light rays incident from the left on a three-disc system. Generated using software from (3).

We examine the three-disc system because gives us some insight into the dynamics of the tetrahedral system. Furthermore, the three-disc system will be much easier to model, and to experimentally control, than a three-dimensional model (e.g., the tetrahedral stack), because the scattering is confined to a plane. For the three-disc system, we can calculate the scattering angle as a function of the impact parameter, which is the shortest distance from a line containing the incoming ray to the geometric center of the array of discs. The three-disc system may also be used to observe a Cantor set (4) for cases in which a ray does not escape from the system (Fig.4.)

An important characteristic of the three-disc system is the ratio of the radius of the disc to the distance between the centers of the
discs. When this ratio decreases, the gaps between the discs become larger, and the number of impacts for an incident light ray decreases on the average. The range of impact parameters leading to a large number of impacts also decreases. We also find that the three-disc system seems to have the highest overall number of impacts from the ray when the ray is directed perpendicular to a gap. This is because more rays from the source can enter the center region of the system and avoid being deflected away from the system.

The tetrahedral system (four mirrored spheres that are stacked into a tetrahedron) is expected to display behavior of the same basic nature as both the two- and three-disc systems. However, because this system has an additional spatial dimension, we expect more complex relationships, similar to the observable fractal patterns shown in Fig. 2. (above).

The computational models of these optical systems are implemented using the Python programming language with the NumPy and PyLab libraries (5). Currently, we have a model for a single ray reflection for a single disc system and are extending the model to a two-disc system. This will be followed by a three-disc system, after which we will model the scattering from the three-dimensional model of the stack of spherical mirrors. The computational models simulate the reflection of light from the cylindrical (or spherical) surfaces. The Python program will calculate the angles of incidence and reflection and will trace the path of the ray as it encounters, and reflects from, the mirror surfaces. We will use this program to plot the relationship between the scattering angle and the impact parameter of the incident ray.

A ray can be mathematically expressed as a line. After choosing an initial starting point and a direction for the line, we can then calculate intersections between the line and the mirror surfaces within our system by mathematically representing the surfaces as circles (or spheres). We know that light cannot pass through the mirrors, so the intersection that is the closest point to the starting point of the ray gives the first point of impact with the disc. Knowing the point of impact and the center point of the disc, we can then construct a linear function for the normal line at the point of impact for the disc. We can subsequently reflect the
ray function with respect to the normal line to construct a linear function for the reflected ray. The point of impact will be used for the initial point of the reflected ray by setting the linear function of the ray equal to the circle function representative of the disc. The program will then repeat this process for the reflected rays, until either there is no intersection between the reflecting ray and any disc within the system (i.e., the ray has left the system), or a maximum number of reflections has occurred. The number of reflections that occur can be stored within the computer model, enabling us to plot the number of reflections as a function of impact parameter. We then change the impact parameter and follow another ray through the system.

If we construct a plot of the number of impacts/collisions versus impact parameter for a three-disc system using the simulation of Hartmann et al. (3), we can see self-similarity occur as we decrease the range of impact parameters, as shown in Figs. 5, and 6.

To date, I have written a program that can trace a single light ray reflecting from a single sphere. A plot of the results is shown in Figure 7., where both x and y represent position. I will

![Figure 5](image1.png)  
**Figure 5:** Number of impacts vs. impact parameter in the range from \( b = 0 \) to \( b = 2.5 \), using the simulation of Hartmann et al. [3] The ratio of diameter to the center-to-center spacing is 6:12.25.

![Figure 6](image2.png)  
**Figure 6:** Number of impacts vs. impact parameter in the range from \( b = 0.75 \) to \( b = 1.25 \), using the simulation of Hartmann et al. [3]. The ratio of the diameter to the center-to-center spacing is 6:12.25.
extend the program to trace several incident rays reflecting from a single sphere. This will allow me to generate a program that can calculate reflections from multiple discs. To do this, I will have to figure out how to keep track of the direction of the ray. Once that is done, I will extend the program into three-dimensional space using the same fundamental ideas and a much more complex algorithm.

II. Experiment

An experimental apparatus of the two- and three-disc system will be constructed by using the corresponding number of mirrored cylinders mounted on top of a rotatable platform. Rotation of the platform enables control of the angle at which the light rays are incident on the array of mirrored cylinders. We will use a diode laser as the light source to approximate a single ray. The diode laser will be mounted on a carriage assembly that rides on a lead screw so that the impact parameter of the incident beam can be controllably varied. A stepper motor controlled by an Arduino will turn the lead screw and thereby translate the carriage assembly. The scattering
angle can be determined using video analysis, by enclosing the apparatus and filling the air within the apparatus with particles, such as dust or water vapor, to make the light path visible. Visible light paths can be recorded, and the recordings can be analyzed to determine the path of the light at given impact parameters. We plan on designing and constructing a three-disc system using reflective cylinders a uniform distance apart, forming an equilateral triangle. This system will have a light source directed at the system. It is likely that we will use video analysis to measure and record the scattering angles, so that we may compare the results of the computational model to observations.

To observe the fractal pattern generated by an illuminated tetrahedral stack of four spherical mirrors, we set up a preliminary experiment. However, we will repeat the experiment with more control over the observing conditions, in which we will then shine three light beams of different color (preferably red, blue, and green, so that the colors can easily be distinguished), into three of the four gaps formed by the stack of spheres. This will be done using non-collimated light sources, together with measurements of the placement and aiming angle of the digital camera used to record the fractal patterns.

The minimal necessary area that the light beams must illuminate is represented by a circle inscribed within an equilateral triangle, with corner points at the center of the spheres, as shown by the blue circle in Figure 8. The radius of this circle is equal to the sphere radius divided by the square root of three:

\[
\frac{r_{\text{sphere}}}{\sqrt{3}} = r_{\text{gap}}
\]

Given that we are using spheres with a radius about 1.25 in., from this we can calculate that our basins will be about \( \frac{1.25}{\sqrt{3}} \) inches in diameter.

To perform the experiment, we will use four mirrored spheres (which are holiday ornaments); three flashlights that produce light beams of red, blue, and green; a cardboard panel; and mechanical supports. The Modern Optics Lab in the Department of Physics and Astronomy at Eastern Michigan University will providing access to a camera and computer. The
Cardboard has four circular holes cut into it, three of which hold the mirrored spheres in place. The fourth hole is aligned with the bottom gap, so that when we elevate the cardboard panel, we may shine a flashlight into the gap. The mechanical supports will be used to fix the light sources in place. We will compare the recorded images of the fractal patterns to the patterns predicted by the computational model.

Possible complications in modeling the patterns can arise from deviations in the shape of the spheres, the degree of collimation of the light beams, and the direction of the light beams. The deviations in the shape of the spheres will be challenging to accurately measure. The collimation and direction of the beams will be measured so that the inputs to the model are known.

We used a Cannon FH-25 camera to record preliminary fractal patterns for the tetrahedral system. There are four different factors that can affect the quality of the recorded pictures, including (1) uniformity of illumination, (2) the divergence and extent of the initial light sources into the system, (3) the pointing direction of the camera towards the system, and (4) the depth of field of the camera. First, the uniformity of illumination is important, because when the light intensity is uniformly distributed over the entire gap region, it is easier to distinguish the different fractals. When the illumination is not uniform over the entire gap, such as from an unfiltered point source, there can be a large falloff.

Figure 8. View normal to a plane containing three sphere centers.
of intensity from the center to the edges of individual fractals. This makes it challenging to accurately identify the boundaries between individual fractals, and dark gaps or regions may start to appear between the fractals. Second, the divergence and extent of the light source can affect the fractal patterns because our chaotic system is very sensitive to the initial conditions, including the initial direction of the light. Consequently, it is important to know the angle at which the light sources are pointed towards the gaps. For some experiments, we want the sources to be pointed perpendicular to the gaps within our systems and directed towards their centers. Third, the pointing direction of the camera affects the observed patterns and determines which portion of the light escaping our system will enter the camera. Consequently, we observe different fractal patterns. We also will have the camera directed perpendicular to a plane containing the center of three of the spheres. Because the images produced by the mirrors appear at different depths, we want to keep our depth of field as large as possible, so that the fractal patterns recorded are not blurred from falling outside the range of focus. However, in order to increase the depth of field, we have to take pictures from farther away, with the consequence that less detail is recorded.

**Figure 9.** shows fractal patterns from our prototype. Two sets of fractal patterns can be observed. These patterns were produced when illuminating three of the four gaps of a tetrahedral stack of spherical mirrors. Current work by the author.

![Figure 9. Fractals formed by chaotic scattering from a tetrahedral stack of spherical mirrors. Current work by the author.](image)
stack of four mirrored spheres. The first fractal pattern is the repeating sphere shape fractal pattern from which we can observe similar spheres repeat until disappearing into bounding lines. These bounding lines also form a complimentary fractal pattern approaching a triangular shape, reminiscent of the gaps formed from the tetrahedral configuration of the spherical mirrors. This triangular shape fractal pattern seems to approach a Sierpinski Gasket (2) fractal pattern when viewed within a two dimensional plane. It is worth noting that although Fig. 2. and Fig. 9. display similar patterns, there is a huge difference between the positioning of these patterns. This is primarily caused by the different vantage points that were used to look into the stack of mirrored spheres.

III. Next Steps

The tasks remaining to complete this project include the construction of the computational models for the two-disc and three-disc systems, the design and construction of the two-disc and three-disc scattering apparatus, and performance of scattering experiments with the new apparatus. I will compare the predictions of the computational models to the data from the experiments. The computational model will then be extended to three dimensions to model the scattering from the tetrahedral stack of mirrored spheres. Scattering experiments for the tetrahedral stack will be repeated under different illumination conditions, such as employing different incident beam directions.

REFERENCES

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The Python programming language was used as well as the language libraries, <http://python.org/>.